Problem 1. Toss a fair coin twice and let $X$ be the number of Heads, and let $Y = 1$ if the first toss is Tails, $Y = 0$ otherwise. Find: (i) the distribution of $X$; (ii) the distribution of $Y$; (iii) $E_X$; (iv) $E_Y$; (v) $\text{Var } X$; (vi) $\text{Var } Y$; (vii) $\text{Cov}(X,Y)$; (viii) $P(X = 2 \mid Y = 0)$.

Solution. (i) $$P(X = 0) = 1/4, \quad P(X = 1) = 1/2, \quad P(X = 2) = 1/4.$$  
(ii) $$P(Y = 0) = 1/2, \quad P(Y = 1) = 1/2.$$  
(iii) $$E_X = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1.$$  
(iv) $$E_Y = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}.$$  
(v) $$E_X^2 = \frac{1}{4} \cdot 0^2 + \frac{1}{2} \cdot 1^2 + \frac{1}{4} \cdot 2^2 = \frac{3}{2},$$  
so $$\text{Var } X = \frac{3}{2} - 1^2 = \frac{1}{2}.$$  
(vi) $$E_Y^2 = \frac{1}{2} \cdot 0^2 + \frac{1}{2} \cdot 1^2 = \frac{1}{2},$$  
so $$\text{Var } Y = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}.$$  
(vii) $\text{Cov}(X,Y) = E(XY) - E_X E_Y$. But $XY = 0$ if first toss Heads ($Y = 0$) or both tosses Tails ($X = 0$), and $XY = 1$ otherwise (first H, second T). So $$E(XY) = 0 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = \frac{1}{4},$$  
and $$E_X = 1, \quad E_Y = \frac{1}{2}.$$  
Thus, $$\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$$  
(viii) $$P(X = 2 \mid Y = 0) = \frac{P(X = 2, Y = 0)}{P(Y = 0)}.$$  
But $$P(X = 2, Y = 0) = P(\text{HH}) = \frac{1}{4},$$  
and $$P(Y = 0) = P(\text{first H}) = \frac{1}{2}.$$  
Therefore, $$P(X = 2 \mid Y = 0) = \frac{1/4}{1/2} = \frac{1}{2}.$$