Review Problems

Problem 1. Suppose you have 7 Danes and 8 Swedes. You need to choose a committee of 3 Danes and 3 Swedes. What is the number of choices:
   (i) without additional constraints;
   (ii) if the committee must contain a chairperson;
   (iii) if the chairperson must be a Swede?

Problem 2. Calculate \( \binom{30}{4} \).

Problem 3. Using Stirling's formula, approximate \( \binom{100}{30} \).

Problem 4. Suppose ten men and twelve women are dancing. How many ways are there to find ten pairs (man-woman), if the ordering of the pairs does not matter?

Problem 5. There are 12 weightlifters: 4 from the US, 4 from Canada, 3 from Russia and 1 from UK. We would like to rank them. How many possible rankings if:
   (i) we give their names in the ranking;
   (ii) we give only their country in the ranking?

Problem 6. Let \( A \) and \( B \) be events such that
\[
P(A \setminus B) = 0.3, \quad P(B) = 0.6, \quad P(A \cup B) = 0.9.
\]
Find \( P(A \cap B) \).

Problem 7. You randomly choose a number between 1 and 1000. What is the probability that it is divisible by 2 and by 5, but not by 10?

Problem 8. Suppose you are throwing three dice.
   (i) What is the probability space?
   (ii) What is the probability that you will have at least one number greater than 4?

Problem 9. In the Wall Street - Russian roulette game, let us change the setup. Suppose the interviewer puts the bullets at two slots so that there is one empty slot between them. Solve the problem for this case.

Problem 10. Toss three coins, blindfolded. Suppose somebody tells you that you tossed at least two Heads. What is the conditional probability that you tosses at least three Heads?

Problem 11. An urn contains 10 black and 10 white balls. Draw five balls: (i) without replacement; (ii) with replacement. What is the probability that all five are white?

Problem 12. Two parents are non-albino. They have three children. What is the probability that all children are non-albino?

Problem 13. Suppose there are 1% of sick people in the population. The test gives wrong result for sick people in 20% cases, and for healthy people in 10% cases. You test
a random person three times. The first and second test give the positive result (that he is sick), but the third test gives the negative result. What is the probability that he is sick?

**Problem 14.** You take a fair die and an unfair die, which has the following probabilities:

\[ P(1) = 0, \ P(2) = P(3) = P(4) = P(5) = \frac{1}{8}, \ P(6) = \frac{1}{2}. \]

You randomly (uniformly) pick a die (not knowing which one you pick) and toss is twice. The first result is 5, the second is 6. What is the probability that this is a fair die?

**Problem 15.** Consider the following random variables \(X\) and \(Y\):

<table>
<thead>
<tr>
<th>(X)</th>
<th>(Y)</th>
<th>Prob.</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1/6</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/6</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1/4</td>
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<tr>
<td>3</td>
<td>0</td>
<td>1/4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1/6</td>
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(i) Find the distribution, expectation and variance for each of these two random variables.
(ii) Find \(\text{Cov}(X,Y)\).
(iii) Find \(\text{Cov}(X,Y^2)\).

**Problem 16.** Suppose you throw a die until the second time you get a six. What is the probability that you need more than three throws?

**Problem 17.** You have ten clients, each will default a mortgage with probability 5%. What is the probability that at least two clients will default a mortgage?

**Problem 18.** You have \(N\) clients, each will default a mortgage with probability \(p\). What is the probability that at least \(k\) clients will default a mortgage, if:

(i) \(N = 1000, \ p = 0.1\%, \ k = 3\);
(ii) \(N = 1000, \ p = 20\%, \ k = 220\)?

Use Poisson or Normal approximation.

**Problem 19.** Suppose the airline is overbooking a Boeing 747-400 with 624 seats (if it contains no first or business class). It sells tickets to \(N\) passengers. The probability that any passenger will not show up is 10%. How large should \(N\) be so that the probability that somebody will be bumped is less than 1%?

**Problem 20.** Let \(X \sim \text{Geo}(2/3)\). Find \(P(X \leq 3)\).

**Problem 21.** An insurance company sells insurance against floods. It has two types of clients. There are \(N_1 = 10000\) low-risk clients, which have a flood with probability \(p_1 = 0.02\%,\) and there are \(N_2 = 1000\) high-risk clients, which have flood with probability \(p_2 = 0.05\%.\) The company pays 100,000$ in case of a flood. What is the value at risk for the confidence level 90%? In other words, how much money should the company amass to avoid bankruptcy with probability at least 90%?

**Problem 22.** The same question as above for \(p_1 = 1\%\) and \(p_2 = 10\%\).
Problem 23. Consider a random variable $X$ with density
\[ p(x) = \begin{cases} 
1, & 0 \leq x \leq 1/2; \\
3/2 - x, & 1/2 \leq x \leq 3/2.
\end{cases} \]
Find $E[X]$, $\text{Var} X$ and $E[X^{-1}]$.

Problem 24. An insurance company sells insurance against floods. It has two types of clients. There are $N_1 = 10000$ low-risk clients, which file claims which are exponentially distributed with mean 1, and there are $N_2 = 1000$ high-risk clients, which file claims which are exponentially distributed with mean 3. What is the value at risk for the confidence level 95%?

Problem 24. Let $X, Y \sim \mathcal{N}(0, 2)$ be independent.
(i) Find $E(X + Y)$;
(ii) Find $E(XY)$;
(iii) Find $E((X + Y)^2)$;
(iv) Find $E(X^2Y^2)$;
(v) Find $\text{Var}(X + Y)$;
(vi) Find $\text{Var}(X - 2Y)$.

Problem 25. Suppose you have $N = 100$ observations. There are 25 zeros, 25 ones, 25 twos and 25 minus ones.
(i) Calculate $\bar{x}$ and $s^2$.
(ii) Test the hypotheses $\mu = 0$ and $\mu = -1$ for $p = .05$, using the normal approximation.
(iii) Find the confidence interval for $\mu$ for the level 95%.

Problem 26. Suppose $Y \sim \text{Exp}(1)$. Find the density of $\log X$. 