Introduction

The author has been teaching and TAing Math 126 (Calculus with Analytic Geometry III) for a while at the University of Washington, Department of Mathematics. This is the last one-quarter course in the basic three-quarter calculus sequence for engineering and science students. We use the textbook *Multivariable Calculus* by James Stewart. For the last part of the course, we use "Taylor Notes" (additional concise lecture notes).

This course covers five topics: analytic geometry (Chapter 12), differential geometry (Chapters 10 and 13), partial derivatives (Chapter 14), double integrals (Chapter 15), Taylor polynomials and series (Taylor Notes). See Appendix for the detailed syllabus.

The main goal of a teaching assistant is to deliver quiz sections (=problem sessions, recitations). These are 50-60 minute classes devoted to homework questions and answers and some additional practice. Sometimes, the instructor asks theirs teaching assistant to read some mini-lectures. Being a teaching assistant, I have exposed a lot of past midterm and final problems on quiz sections. (There are two midterms and one final exam.) I used to prepare a handout of five or six problems related to the most recent studied topic, and discuss them with students after all homework question had been answered. This was not a group work, but rather an individual work. After quiz sections were finished, I e-mailed solutions to the students, and posted them on the class web page.

It was essential to provide students with solutions, since they were eager to see the correct way of solving these problems, but the departmental Math 126 website:

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http://www.math.washington.edu/m126/
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does not contain all solutions to old midterms and finals. All problems are taken from one of these archives:

**Departmental Math 126 website**, containing three archives:
- Old 1st midterms: http://www.math.washington.edu/m126/midterms/midterm1.php
- Old 2st midterms: http://www.math.washington.edu/m126/midterms/midterm2.php
- Old Finals: http://www.math.washington.edu/m126/finals/final.php

My own midterm archive:
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Dr Matthew Conroy’s midterm archive:
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http://www.math.washington.edu/conroy/m126-general/exams/
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Dr Andrew Loveless’ midterm archive:
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The book is organized as follows. The first part is compiled of the lectures which I have read as an instructor. They are edited and reformatted a bit. The second part of this book contains past exam problems. Each problem contains a reference, say: [Midterm 1, Conroy, Spring 2007, 2]. This is a second problem in the first midterm made by Dr Conroy in Spring Quarter 2007. All problems are classified in five groups, according to their topic. The third part of the book contains all solutions. The appendix contains a detailed syllabus of this course.
Lectures

1. Three-Dimensional Coordinate System

On the plane, we can represent any point by a pair of two Cartesian coordinates \((x, y)\), which are defined by two perpendicular axes: the \(x\)-axis and the \(y\)-axis. In the space, we have three Cartesian coordinates \((x, y, z)\) and three mutually perpendicular axes: \(x\), \(y\) and \(z\)-axis. The direction of the \(z\)-axis is determined by the right-hand rule: if you curl the fingers of your right hand from the \(x\)-axis to the \(y\)-axis, provided the hand is centered at the origin (the point of their intersection), then the thumb points to the direction of the \(z\)-axis.

There are three ways to interpret three-dimensional Cartesian coordinates.

1. Suppose \(P\) is any point in the space. Put the perpendicular \(PQ\) down to the \(x\)-axis. Then the point \(Q\) lies on the \(x\)-axis.
   - If it lies on the positive side, then the distance between \(O\) (the origin) and \(Q\) is called the \(x\)-coordinate of \(P\).
   - If \(Q\) lies on the negative side, then minus this distance is called the \(x\)-coordinate of \(P\).
   - If \(Q\) coincides with \(O\), then this coordinate is equal to zero and the point \(P\) lies on the \(yz\)-coordinate plane.

2. Also, we can view the \(x\)-coordinate of \(P\) as the signed distance from \(P\) to this \(yz\)-coordinate plane. The \(y\)- and \(z\)-coordinates are defined similarly.

3. Also, suppose we have a triple \((x_0, y_0, z_0)\). Start from the origin \(O\). Go along the \(x\)-axis by the distance \(x_0\). This means:
   - if \(x_0 > 0\) go in the direction of the \(x\)-axis and cover the distance \(x_0\);
   - if \(x_0 < 0\) go in the opposite direction and cover the distance \(|x_0|\);
   - if \(x_0 = 0\) just stay still.

   Reach some point \(Q\), then go along the \(y\)-axis by the distance \(y_0\), reach \(R\), then go along the \(z\)-axis by the distance \(z_0\). You will reach the point \(P\) with coordinates \((x_0, y_0, z_0)\).

The set of all triples \((x, y, z)\) of real numbers, or, in other words, the set of all points in the space will be denoted by \(\mathbb{R}^3\). Just like the set of all pairs \((x, y)\) of real numbers, i.e. the set of all points on the plane will be denoted by \(\mathbb{R}^2\).

Example. In \(\mathbb{R}^3\), \(z = 0\) is the \(xy\)-coordinate plane, \(y = 5\) is the plane which is parallel to the \(xz\)-coordinate plane and 5 units away from it, in the direction of the \(y\)-axis. \(y = 5\), \(z = 3\) is a line parallel to the \(x\)-axis and 5 units away from the \(xz\)-plane and 3 units away from the \(xy\)-plane.

The distance formula. In \(\mathbb{R}^2\), the distance between the two points \((x_1, y_1)\) and \((x_2, y_2)\) is \(\sqrt{(x_2-x_1)^2+(y_1-y_2)^2}\). Similarly, in \(\mathbb{R}^3\), the distance between \(P_1 = (x_1, y_1, z_1)\) and \(P_2 = (x_2, y_2, z_2)\) is

\[
    d := \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.
\]

Indeed, assume for simplicity \(P_2 = O\), the origin. Let us move from the origin to \(P = P_1\), as described above. Then \(|OQ| = |x_1|\), \(|QR| = |y_1|\), and the triangle \(OQR\) has right angle \(Q\). By the Pythagorean theorem, \(|OR|^2 = |OQ|^2 + |QR|^2 = |x_1|^2 + |y_1|^2\). Similarly, the triangle \(ORA\) has right angle \(R\), and \(|RP| = |z_1|\). So \(|OA|^2 = |OR|^2 + |RA|^2 = |x_1|^2 + |y_1|^2 + |z_1|^2 = x_1^2 + y_1^2 + z_1^2\).

Example. The distance between \((2, -1, 6)\) and \((3, -3, 8)\) is

\[
    \sqrt{(2-3)^2+(-1-(-3))^2+(6-8)^2} = \sqrt{1 + 4 + 4} = 3.
\]
Equations of spheres. A sphere with radius $r$ and center $P_0(x_0, y_0, z_0)$ is the surface which consists of all points $P$ such that the distance from $P$ to $P_0$ is $r$. If $P = (x, y, z)$, then its equation is
\[
\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r \iff (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.
\]
This is similar to the equation of the circle on $\mathbb{R}^2$ with center $(x_0, y_0)$ and radius $r$: $(x - x_0)^2 + (y - y_0)^2 = r^2$.

**Example.** The unit sphere (i.e. centered at the origin with radius 1) has the equation $x^2 + y^2 + z^2 = 1$.

**Example.** $x^2 - 2x + y^2 + 4y + z^2 = 5$ also represents a sphere: let us complete the squares,
\[
x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 = 10 \iff (x - 1)^2 + (y + 2)^2 + z^2 = \sqrt{10}^2.
\]
This is a sphere with radius $\sqrt{10}$ centered at $(1, -2, 0)$.
2. Vectors in Three Dimensions

**Definition of vectors.** The term vector is used to indicate a quantity (e.g. velocity or force) with both magnitude and direction. A quantity which have only magnitude without direction (i.e. real numbers) is called a scalar. We denote a vector by printing a letter in boldface: \( \mathbf{v} \) or by an arrow: \( \overrightarrow{v} \).

Suppose a particle has moved from the point \( A \) to the point \( B \). The displacement vector \( \mathbf{v} = \overrightarrow{AB} \) has initial point \( A \) and terminal point \( B \). When we write a vector as a combination of two letters, the first letter denotes the initial point and the second letter denotes the terminal point.

If two vectors \( \mathbf{u} \) and \( \mathbf{v} \) have the same direction and length, they are called equal: \( \mathbf{u} = \mathbf{v} \). The zero vector \( \mathbf{0} = \overrightarrow{AA} \) has length 0 and no specific direction.

The magnitude, or length, of the vector \( \mathbf{v} = \overrightarrow{AB} \) is the distance from \( A \) to \( B \). It is denoted by \( |\mathbf{v}| \).

**Addition.** If we have two vectors \( \mathbf{u} \) and \( \mathbf{v} \), how to add them?

**Triangle Rule.** Suppose a particle moved from \( A \) to \( B \) so that \( \mathbf{u} = \overrightarrow{AB} \) and then moved from \( B \) to \( C \) so that \( \mathbf{v} = \overrightarrow{BC} \). Then the ”sum” of these movements is the movement from \( A \) to \( C \), so \( \mathbf{u} + \mathbf{v} = \overrightarrow{AC} \) by definition.

**Parallelogram Rule.** Draw \( \mathbf{u} = \overrightarrow{AB} \) and \( \mathbf{v} = \overrightarrow{AD} \) from the same initial point \( A \) and construct a parallelogram based on these vectors: a parallelogram \( ABCD \). Then, by definition, \( \overrightarrow{AC} = \mathbf{u} + \mathbf{v} \).

These rules are equivalent, since \( \overrightarrow{BC} = \overrightarrow{AD} = \mathbf{v} \).

**Multiplication by a scalar.** The idea is: \( 2\mathbf{v} = \mathbf{v} + \mathbf{v} \) has the same direction and its magnitude is the doubled magnitude of \( \mathbf{v} \).

In general, let \( c \) be a scalar and let \( \mathbf{v} \) be a vector. Then \( \mathbf{u} = c\mathbf{v} \) has magnitude \( |\mathbf{u}| = |c||\mathbf{v}| \), and its direction is the same as the direction of \( \mathbf{v} \) if \( c > 0 \), and is opposite if \( c < 0 \). If \( c = 0 \), then \( c\mathbf{v} = \mathbf{0} \). Also, if \( \mathbf{v} = \mathbf{0} \), then \( c\mathbf{v} = \mathbf{0} \).

Important example: if \( \overrightarrow{AB} = \mathbf{v} \), then \( \overrightarrow{BA} = -\mathbf{v} = (-1) \cdot \mathbf{v} \). The magnitude of \( -\mathbf{v} \) is the same as the magnitude of \( \mathbf{v} \), and the length is opposite.

**Subtraction.** For any vectors \( \mathbf{u} \) and \( \mathbf{v} \), \( \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) \). Draw the vectors \( \mathbf{u}, \mathbf{v} \) from the same initial point \( A \): \( \mathbf{u} = \overrightarrow{AB}, \mathbf{v} = \overrightarrow{AC} \). Then \( -\mathbf{v} = \overrightarrow{CA} \) and \( \mathbf{u} - \mathbf{v} = \overrightarrow{AB} + \overrightarrow{CA} = \overrightarrow{CB} \).

**Components.** We can write every vector as three real numbers: suppose \( \mathbf{u} = \overrightarrow{OA} \), where \( O \) is the origin and \( A = (x_0, y_0, z_0) \) is a point in space. Then, by definition, \( \mathbf{u} = \langle x_0, y_0, z_0 \rangle \). These numbers \( x_0, y_0, z_0 \) are called the components of \( \mathbf{u} \). So any vector is defined by its components, and, conversely, any three real numbers are components of some vector.

If \( \mathbf{u} = \overrightarrow{AB} \), where \( A = (x_1, y_1, z_1) \) and \( B = (x_2, y_2, z_2) \), then \( \mathbf{u} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \).

Indeed, suppose \( \mathbf{u} = \langle \alpha, \beta, \gamma \rangle \). Then \( \mathbf{B} = (x_1 + \alpha, y_1 + \beta, z_1 + \gamma) \), but at the same time \( \mathbf{B} = (x_2, y_2, z_2) \). Comparing the coordinates of \( \mathbf{B} \), we get:

\[
\begin{align*}
x_1 + \alpha &= x_2, \\
y_1 + \beta &= y_2, \\
z_1 + \gamma &= z_2,
\end{align*}
\]

so we get:

\[
\alpha = x_2 - x_1, \quad \beta = y_2 - y_1, \quad \gamma = z_2 - z_1.
\]

**Example.** If \( A = (0, -2, 3) \) and \( B = (1, -3, 6) \), then \( \overrightarrow{AB} = \langle 1, -1, 3 \rangle \).

**Applications of components.** If \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \), then \( \mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \). Indeed, suppose \( \mathbf{u} = \overrightarrow{OA} \) and \( \mathbf{v} = \overrightarrow{OB} \). Then \( \mathbf{A} = (u_1, u_2, u_3) \), \( \mathbf{B} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \) and \( \mathbf{u} + \mathbf{v} = \overrightarrow{OB} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \).
Also, if \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \), then \( c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle \). (See the book for detailed explanation.) In particular, \( -\mathbf{u} = \langle -u_1, -u_2, -u_3 \rangle \). And \( \mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u}) = \langle v_1 - u_1, v_2 - u_2, v_3 - u_3 \rangle \). Thus, to add or subtract vectors, we just add or subtract their components.

The magnitude \( |\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} \). Indeed, let \( \mathbf{u} = \overline{OA} \). Then \( A = (u_1, u_2, u_3) \). And the distance between \( O \) and \( A \) (which is the magnitude \( |\mathbf{u}| \) of \( \mathbf{u} \)) is \( \sqrt{u_1^2 + u_2^2 + u_3^2} \).

**Example.** For \( \mathbf{u} = \langle 1, 2, 2 \rangle \), we have:

\[
|\mathbf{u}| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3.
\]

The vectors \( \mathbf{i} = \langle 1, 0, 0 \rangle \), \( \mathbf{j} = \langle 0, 1, 0 \rangle \) and \( \mathbf{k} = \langle 0, 0, 1 \rangle \) play a special role: any vector can be represented as a linear combination of these three vectors. More precisely, if \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \), then \( \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \).

A **unit vector** is a vector with length 1. E.g. \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are unit vectors. If \( \mathbf{a} \neq 0 \), then \( \mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} \) is a unit vector which has the same direction as \( \mathbf{a} \), because its length is \( \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1 \). Say, the unit vector in the direction of \( \mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} \) is \( \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \), because \( |\mathbf{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3 \).

**Applications in physics.** The magnitude of a velocity is called **speed**. Suppose a wind is blowing from \( N45^\circ W \) at 50km/h. This means the direction is 45\(^\circ\) west from the northern direction. A pilot is steering a plane in \( N60^\circ E \) at an **airspeed** (speed in still air) 250km/h. The **true course**, or **track**, is the direction of the resultant of the velocity vectors of the plane and the wind. The **ground speed** is the magnitude of the resultant.

Here, the velocity of the wind is

\[
\mathbf{u} = \langle 50 \cos 45^\circ, -50 \sin 45^\circ \rangle = \langle 50/\sqrt{2}, -50/\sqrt{2} \rangle = \langle 25\sqrt{2}, -25\sqrt{2} \rangle,
\]
and the airspeed is

\[
\mathbf{v} = \langle 250 \sin 60^\circ, 250 \cos 60^\circ \rangle = \langle 250/2, 250 \cdot \sqrt{3}/2 \rangle = \langle 125, 125\sqrt{3} \rangle.
\]
So the resultant is

\[
\mathbf{w} = \mathbf{u} + \mathbf{v} = \langle 25\sqrt{2} + 125, -25\sqrt{2} + 125\sqrt{3} \rangle \approx \langle 160, 181 \rangle.
\]
Thus, \( |\mathbf{w}| = \sqrt{160^2 + 181^2} = 242 \) - the ground speed, and the true course is \( N41^\circ E \), because \( 41^\circ \approx \arctan(160/181) \).
3. Dot Product

If \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) and \( \mathbf{b} = \langle b_1, b_2, b_3 \rangle \), then their dot product \( \mathbf{a} \cdot \mathbf{b} \) is a number given by

\[
\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

For two-dimensional vectors \( \mathbf{a} = \langle a_1, a_2 \rangle \) and \( \mathbf{b} = \langle b_1, b_2 \rangle \), this is just \( a_1 b_1 + a_2 b_2 \).

**Example.** \( \mathbf{a} = \langle 3, -1 \rangle \), \( \mathbf{b} = \langle 1, 2 \rangle \) \( \Rightarrow \) \( \mathbf{a} \cdot \mathbf{b} = 3 \cdot 1 + (-1) \cdot 2 = 1. \)

\( \mathbf{a} = \langle 1, -3, 0 \rangle \), \( \mathbf{b} = \langle 6, 4, 3 \rangle \) \( \Rightarrow \) \( \mathbf{a} \cdot \mathbf{b} = 1 \cdot 6 + (-3) \cdot 4 + 0 \cdot 3 = -6. \)

**Properties.** If \( \mathbf{a} \), \( \mathbf{b} \), \( \mathbf{c} \) are vectors and \( \lambda \) is a scalar, then

1. \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \), dot square of \( \mathbf{a} \);
2. \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \);
3. \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \);
4. \( (\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda (\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b}) \);
5. \( \mathbf{0} \cdot \mathbf{a} = 0. \)

Indeed, let us prove, say, 1: \( \mathbf{a} \cdot \mathbf{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2. \)

3. \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \langle a_1, a_2, a_3 \rangle \cdot (\langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle) = \langle a_1 b_1 + a_2 b_2 + a_3 b_3 \rangle + (a_1 c_1 + a_2 c_2 + a_3 c_3) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \)

Suppose \( \theta \) is the angle between two nonzero vectors \( \mathbf{a} \) and \( \mathbf{b} \): \( 0 \leq \theta \leq \pi. \) If \( \theta = 0 \), they have the same direction; if \( \theta = \pi/2 \), they are orthogonal, i.e. perpendicular; if \( \theta = \pi \), they have opposite directions. Then

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.
\]

In fact, physicists consider this as a definition of dot product.

**Proof.** Let \( \mathbf{OA} = \mathbf{a}, \mathbf{OB} = \mathbf{b} \). Then \( \mathbf{AB} = \mathbf{b} - \mathbf{a} \), apply the Law of Cosines to the triangle \( OAB \):

\[
|\mathbf{OA}|^2 + |\mathbf{OB}|^2 - 2|\mathbf{OA}||\mathbf{OB}| \cos \theta = |\mathbf{AB}|^2 \Rightarrow |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta = |\mathbf{b} - \mathbf{a}|^2.
\]

But

\[
|\mathbf{b} - \mathbf{a}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{b} \cdot (\mathbf{b} - \mathbf{a}) - \mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} = |\mathbf{b}|^2 + |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b}.
\]

Therefore, \( |\mathbf{a}|^2 + |\mathbf{b}|^2 \) cancels out, and we have

\[
-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}| \cos \theta.
\]

Cross out \(-2\) and finish the proof.

**Example.** If \( \mathbf{a} = \langle 1, 2, -1 \rangle \) and \( \mathbf{b} = \langle 3, 0, 1 \rangle \), then

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{\sqrt{6}\sqrt{10}} \approx 0.258, \quad \theta \approx 75^\circ.
\]

The angle \( \theta \) is acute (\( \theta < \pi/2 \)) iff \( \cos \theta > 0 \iff \mathbf{a} \cdot \mathbf{b} > 0. \)

The angle \( \theta \) is obtuse (\( \theta > \pi/2 \)) iff \( \cos \theta < 0 \iff \mathbf{a} \cdot \mathbf{b} < 0. \)

The angle \( \theta \) is right (\( \theta = \pi/2 \)) iff \( \cos \theta = 0 \iff \mathbf{a} \cdot \mathbf{b} = 0. \)

**Projections.** Let \( \mathbf{a} = \overrightarrow{PQ} \neq \mathbf{0}. \) Let \( \mathbf{b} = \overrightarrow{PR}. \) If \( S \) is the foot of the perpendicular from \( R \) to the line containing \( \overrightarrow{PQ}, \) then \( \overrightarrow{PS} \) is called the vector projection of \( \mathbf{b} \) onto \( \mathbf{a} \) and is denoted by \( \text{proj}_a \mathbf{b}. \) It is “a shadow of \( b \).”

The scalar projection of \( \mathbf{b} \) onto \( \mathbf{a} \) is defined as the signed magnitude of \( \overrightarrow{PS}, \) i.e. \( |\mathbf{b}| \cos \theta, \) where \( \theta \) is again the angle between \( \mathbf{a} \) and \( \mathbf{b}. \) "Signed" means it can be positive or negative, depending
on whether $PQ$ and $PS$ have the same direction or opposite directions, i.e. whether $\theta$ is acute or obtuse. The scalar projection is denoted by $\text{comp}_a b$.

Note that $a \cdot b = |a||b| \cos \theta = |a| \text{comp}_a b$. Therefore,

$$\text{comp}_a b = \frac{a \cdot b}{|a|}.$$ 

And if $u = (|a|)^{-1}a$ is the unit vector with the same direction as $a$, then

$$\text{proj}_a b = (\text{comp}_a b) u = \frac{a \cdot b}{|a|} \cdot \frac{a}{|a|} = \frac{a \cdot b}{|a|^2} a.$$ 

**Example.** For $a = <1, 2, -1>$ and $b = <3, 0, 1>$, we have: $a \cdot b = 2$, $|a| = \sqrt{6}$, so $\text{comp}_a b = 2/\sqrt{6}$, $\text{proj}_a b = \frac{2}{\sqrt{6}} <1, 2, -1> = \frac{1}{3}, \frac{2}{3}, -\frac{1}{3}>.$

**Example.** For $a = <a_1, a_2, a_3>$, we have: $a \cdot i = a_1$, $a \cdot j = a_2$, $a \cdot k = a_3$, and $|i| = |j| = |k| = 1$. Therefore,

$$\text{comp}_i a = a_1, \text{comp}_j a = a_2, \text{comp}_k a = a_3; \text{proj}_i a = a_1 i, \text{proj}_j a = a_2 j, \text{proj}_k a = a_3 k.$$ 

**Applications in physics.** If a body made a displacement $d = AB$ from $A$ to $B$, and a constant (i.e. not changing with time) force $F$ was acting on it, then the work done by the force $F$ is $dW = d \cdot F$. A body of mass $m$ moving with velocity $v$ (i.e. speed $v = |v|$) has kinetic energy $E = mv^2/2$. If $F$ is a total force (i.e. sum of all forces) acting on a body which made a displacement $d$, then the change of its kinetic energy is equal to the work done by $F$, i.e. to $dW = d \cdot F$.

**Example.** Suppose a body of mass $m$ starts moving on a slope with angle $\alpha$ from the height $h$. The initial speed $v_0 = 0$ is zero. What speed $v$ does it gain at the end?

Two forces are acting on this body: the gravity $mg$ and the reaction $N$. The displacement $d = AB$ is orthogonal to $N$, so $d \cdot N = 0$. Also, $d \cdot (mg) = mgh$, because $d \cdot g = |g| \text{comp}_g d = gh$. Thus, $dW = mgh$, and this is equal to the change in kinetic energy. The initial kinetic energy is zero, so the terminal kinetic energy is $mv^2/2 = mgh$, and $v = \sqrt{2gh}$. Note: it does not depend on $m$ and on the angle $\alpha$!
4. Cross Product

Find a nonzero vector \( \mathbf{v} = \langle x, y, z \rangle \) orthogonal to \( \mathbf{a} = \langle 1, -1, 0 \rangle \) and \( \mathbf{b} = \langle 1, 0, 2 \rangle \). We need:

\[
0 = \mathbf{v} \cdot \mathbf{a} = x - y, \quad 0 = \mathbf{v} \cdot \mathbf{b} = x + 2z \Rightarrow x = y = -2z.
\]

Say, let \( z = 1 \). Then \( x = y = -2 \). General setting: find \( \mathbf{v} = \langle x, y, z \rangle \) orthogonal to \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) and \( \mathbf{b} = \langle b_1, b_2, b_3 \rangle \). We need:

\[
a_1x + a_2y + a_3z = b_1x + b_2y + b_3z = 0.
\]

A solution of this is

\[
x = a_2b_3 - a_3b_2, \quad y = a_3b_1 - a_1b_3, \quad z = a_1b_2 - a_2b_1.
\]

This vector \( \mathbf{v} \) is denoted by \( \mathbf{a} \times \mathbf{b} \) and is called the cross product of \( \mathbf{a} \) and \( \mathbf{b} \). Unlike the dot product, here the product of two vectors is again a vector, not a scalar.

**Example.** 1. For \( \mathbf{a} = \langle 1, 3, 4 \rangle \) and \( \mathbf{b} = \langle 2, 7, -5 \rangle \), the cross product is equal to \(-43 \mathbf{i} + 13 \mathbf{j} + \mathbf{k} \).

2. \( \langle 1, -1, 0 \rangle \times \langle 1, 0, 2 \rangle = \langle -2, -2, 1 \rangle \).

There are two possible (mutually opposite) directions orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \) (if the latter are nonzero vectors). Which one is the direction of the cross product? It is defined by the right-hand rule: if the fingers of your right hand are curled in the direction of rotation from \( \mathbf{a} \) to \( \mathbf{b} \) (through an angle less than \( \pi = 180^\circ \)), then your thumb points in the direction of \( \mathbf{a} \times \mathbf{b} \).

What is the length of the cross product? After algebraic manipulations, such as expanding and collecting brackets, we get:

\[
|\mathbf{a} \times \mathbf{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) -
\]

\[
(a_1b_1 + a_2b_2 + a_3b_3)^2 = |\mathbf{a}||\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}||\mathbf{b}|^2 - |\mathbf{a}||\mathbf{b}||\mathbf{a}|\cos \theta = |\mathbf{a}||\mathbf{b}||\mathbf{a}| \sin \theta.
\]

where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \). Since \( 0 \leq \theta \leq \pi \), we have: \( \sin \theta \geq 0 \), and \( \sqrt{\sin^2 \theta} = \sin \theta \). Thus,

\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta.
\]

Now we know the magnitude and the direction of the cross product, which uniquely define it. In fact, that’s how physicists define the cross product.

Two vectors \( \mathbf{a} \) and \( \mathbf{b} \) are parallel (with the same OR opposite directions) iff \( \theta = 0 \) or \( \theta = \pi \), i.e. iff \( \sin \theta = 0 \leftrightarrow |\mathbf{a} \times \mathbf{b}| = 0 \leftrightarrow \mathbf{a} \times \mathbf{b} = 0 \). E.g. \( \mathbf{a} = \langle 1, 3, 4 \rangle \) and \( \mathbf{b} = \langle 2, 7, -5 \rangle \) have nonzero cross product (calculated above), so they are not parallel. Actually, it is more or less obvious because their components are not proportional, but the cross product is another way to figure this out.

If \( \mathbf{a} = \overline{PQ} \) and \( \mathbf{b} = \overline{PR} \) are two nonzero vectors, then there is a parallelogramm \( PQSR \) based on \( \mathbf{a} \) and \( \mathbf{b} \). It has side \(|PQ| = ||\mathbf{a}|\) and height \(|\mathbf{b}| \sin \theta\), and area \(|\mathbf{a}||\mathbf{b}| \sin \theta = |\mathbf{a} \times \mathbf{b}|\). This is the geometric meaning of the cross product. The area of the triangle \( PQR \) is \((1/2)|\mathbf{a} \times \mathbf{b}|\).

**Example.** Find the area of the triangle \( PQR \), where \( P = (0, 0, 0) \), \( Q = (1, -1, 0) \), \( R = (1, 0, 2) \). We have: \( \overline{PQ} = \langle 1, 1, 0 \rangle \), \( \overline{PR} = \langle 1, 0, 2 \rangle \), so \( \overline{PQ} \times \overline{PR} = \langle -2, -2, 1 \rangle \), its magnitude is \( \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3 \), and the area of this triangle is equal to \( 3/2 \).

**Properties.** If \( \mathbf{a} \), \( \mathbf{b} \), \( \mathbf{c} \) are vectors in three dimensions and \( \lambda \) is a scalar, then

1. \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \);
2. \( \mathbf{a} \times \mathbf{a} = \mathbf{0} \);
3. \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \);
4. \((\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \);
5. \((c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b}) \);
6. \(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \);
7. \(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \).

Each of them can be proved by writing everything out in components. E.g. let us prove 6:

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1),
\]

\[
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3.
\]

The right-hand sides are equal: just expand the brackets. Note that the cross product is neither commutative: \( \mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a} \), nor associative: \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \).

**Applications in physics.** Suppose a force \( \mathbf{F} \) acts on a rigid body at a point \( P \) with \( \mathbf{r} = \overrightarrow{OP} \), where \( O \) is the origin. The **torque** (relative to the origin) \( \tau := \mathbf{r} \times \mathbf{F} \). If the body is in equilibrium, then the sum of all torques is equal to \( \mathbf{0} \).
5. Lines and Planes

Lines. A line \( l \) in \( \mathbb{R}^3 \) is defined by any point \( P_0 = (x_0, y_0, z_0) \) on it and its direction, represented by a directional vector \( \mathbf{v} = < a, b, c > \). A point \( P = (x, y, z) \) lies on \( l \) iff \( \overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0 = < x - x_0, y - y_0, z - z_0 > \) is parallel to \( \mathbf{v} \), where \( \mathbf{r} = \overrightarrow{OP} \), \( \mathbf{r}_0 = \overrightarrow{OP}_0 \), i.e. iff for some real number \( t \) we have:
\[
\mathbf{r} - \mathbf{r}_0 = t \mathbf{v}.
\]
This is called the vector equation of \( l \). The scalar \( t \) is called the parameter. Rewrite this as
\[
x = x_0 + at, \ y = y_0 + bt, \ z = z_0 + ct.
\]
These are parametric equations of \( l \). Each value of \( t \) gives a point on \( l \), and vice versa.

Example. \( P_0 = (1,2,3) \), \( \mathbf{v} = <-1,2,0> \). Then we have:
\[
x = 1 - t, \ y = 2 + 2t, \ z = 3.
\]
Say, \( t = 1 \Rightarrow (x, y, z) = (0, 4, 3) \), \( t = -1 \Rightarrow (x, y, z) = (2, 0, 3) \). This line intersects the \( xyz \)-plane at \( 2 + 2t = 0 \Rightarrow t = -1 \), i.e. at the point \( (2, 0, 3) \). We can change \( P_0 \) to any other point on \( l \), i.e. \( P_1(0,4,3) \); then we have:
\[
x = -t, \ y = 4 + 2t, \ z = 3.
\]
These are also parametric equations for \( l \). Also, we can change \( \mathbf{v} \) to any nonzero vector parallel to \( \mathbf{v} \), say \(-2\mathbf{v} = <-2, -4, 0>\), so that
\[
x = 1 + 2t, \ y = 2 - 4t, \ z = 3.
\]

Skewness. In \( \mathbb{R}^2 \), two lines either intersect or are parallel. In \( \mathbb{R}^3 \), there is another alternative: they are skew, i.e. do not lie on the same plane. Indeed, if they are parallel or intersect, then they lie on a common plane.

Example. Let \( l_1 \) be given by \( x = 1 + t, \ y = -2 + 3t, \ z = 4 - t \). Let \( l_2 \) be given by \( x = 2s, \ y = 3 + s, \ z = -3 + 4s \). They are not parallel because their directional vectors \( <1,3,-1>\) and \( <2,1,4>\) are not parallel. (Their components are not proportional; also, you can verify this by taking their cross product and observing it is not zero.) Let us find their point of intersection. It satisfies the system of equations
\[
1 + t = 2s, \ -2 + 3t = 3 + s, \ 4 - t = -3 + 4s.
\]
We have: \( t = 2s - 1 \) from the first equation. Plug it into the second equation: \( -2 + 3(2s - 1) = 3 + s, \ 4 - (2s - 1) = -3 + 4s \), i.e. \( 6s - 5 = 3 + s, \ 5 - 2s = -3 + 4s \), i.e. \( 5s = 8, \ 6s = 8, \ s = 8/5, \ s = 4/3 \). So this system does not have a solution, and the lines actually do not intersect, i.e. they are skew.

Planes. A plane does not have a “directional vector”, since a single vector parallel to a plane is not enough to grasp its “direction”. However, a normal vector, i.e. a nonzero vector orthogonal to the plane is enough. Let \( \mathbf{n} = < a, b, c > \) be a normal vector, and let \( P_0 = (x_0, y_0, z_0) \) be any point on the plane \( \pi \). Then \( P = (x, y, z) \in \pi \) iff \( \overrightarrow{P_0P} \perp \mathbf{n} \Leftrightarrow \overrightarrow{P_0P} \cdot \mathbf{n} = 0 \). If \( \mathbf{r} = \overrightarrow{OP}, \mathbf{r}_0 = \overrightarrow{OP}_0 \), then we have:
\[
(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \Leftrightarrow \mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n}.
\]
This is a vector equation of \( \pi \). Rewrite it in coordinate form:
\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.
\]
Example. \( n = < -1, 2, 0 >, P_0 = (1, 2, 3) \), then we have:

\[-1(x - 1) + 2(y - 2) + 0(z - 3) = 0 \iff -x + 1 + 2y - 4 = 0 \iff 2y - x = 3.\]

We can also take a mutliple of \( n \), and change point \( P_0 \) to some other point on the plane, e.g. to \((0,3/2,1)\), and the equation will remain the same. Lack of the \( z \)-coordinate means this plane is parallel to the \( z \)-axis. Indeed, the directional vector of the \( z \)-axis, i.e. \( k \), is orthogonal to \( n \):

\[ k \cdot n = 0, \text{ and so } k \text{ is parallel to this plane.} \]

The \( x \)-intercept (i.e. intersection with the \( x \)-axis) : let \( y = z = 0 \), then \(-x = 3 \Rightarrow x = -3\). The \( y \)-intercept: \( x = z = 0 \Rightarrow 2y = 3 \Rightarrow y = 3/2 \).

Example. The plane that passes through \( P = (0,0,0) \), \( Q = (1,-1,0) \), \( R = (1,0,2) \) has a normal vector: \( n = \overrightarrow{PQ} \times \overrightarrow{PR} = < 1, -1, 0 > \times < 1, 0, 2 > = < -2, -2, 1 > \), and the equation is:

\[ -(2)(x - 0) + (-2)(y - 0) + 1(z - 0) = 0 \iff -2x - 2y + z = 0. \]

Example. The line \( x = 1 - t, y = 2 + 2t, z = 3 \) intersects the plane \( 2y - x = 3 \) at \( 2(2 + 2t) - (1 - t) = 3 \iff 5t + 3 = 3 \iff t = 0 \), i.e. at \((x,y,z) = (1,2,3)\).

Example. Consider the following planes \( x + y + z = 1 \) and \( y - 2z = 0 \). Let \( n_1 = < 1, 1, 1 > \), \( n_2 = < 0, 1, -2 > \) be their normal vectors. Let us find the line of intersection of these two planes. First, its directional vector lies on each of these planes; therefore, it is orthogonal to both normal vectors, and we can take their cross product as the directional vector. (Recall that the directional vector is defined up to a multiplication by a nonzero scalar.) So \( v = n_1 \times n_2 = < -3, 2, 1 > \).

Then, find any point on this line. The system of two equations with three variables has infinitely many solutions, because there are “too many” variables. Let one of the variables be equal to zero; then you will have equal number of equations and variables. E.g. let \( z = 0 \); then \( y = 0 \) and \( x = 1 \), so \( P_0(1,0,0) \) lies on this line, and its equation:

\[ x = 1 - 3t, \ y = 2t, \ z = t. \]
6. Conic Sections and Quadric Surfaces

**Conic sections.** Linear equations in \(\mathbb{R}^2\) represent lines: \(ax + by = c\). Quadratic equations represent hyperbolas, parabolas and ellipses, which are called **conic sections** because they can be obtained as an intersection of different planes with the cone \(x^2 + y^2 = z^2\).

Fix two points: \(A = (c, 0)\) and \(B = (-c, 0)\). An **ellipse** is the set of all points \(P(x, y)\) such that \(|PA| + |PB| = 2a\), where \(a > c\) is fixed. We have: \(\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a\). After algebraic manipulations, we get:

\[
x^2/a^2 + y^2/b^2 = 1, \quad b^2 = a^2 - c^2.
\]

The points \(A\) and \(B\) are foci of the ellipse, \(a\) is its **major half-axis** and \(b\) is its **minor half-axis**. \(e = c/a = \sqrt{a^2 - b^2}/a\) is its **eccentricity**. We always have \(e < 1\), and if \(e = 0\), then \(a = b\), \(c = 0\) and this is simply a circle centered at the origin with radius \(a\). The Earth’s orbit is an ellipse with the Sun in a focus and \(2a = 2.99 \cdot 10^8\), \(e = 0.017\), so it is almost a circle.

A **parabola** is a set of points \(P(x, y)\) equidistant from the line \(x = -p\) (the **directrix**) and the point \(F(p, 0)\) (its **focus**). We have:

\[
\sqrt{(x-p)^2 + y^2} = |x + p| \iff (x-p)^2 + y^2 = (x+p)^2 \iff y^2 = 4px.
\]

A **hyperbola** is defined similarly to the ellipse: \(|PB| - |PA| = 2a\), \(a < c\). It has equation

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad b^2 = c^2 - a^2.
\]

Its **eccentricity** is \(e = c/a = \sqrt{a^2 + b^2}/a > 1\). If \(a = b = 1\), we have: \(x^2 - y^2 = 1\), which after rotation by \(45^\circ\) can become \(xy = 1, \ y = 1/x\ - a\) standard hyperbola.

We can switch the roles of \(x\) and \(y\), shift these curves. E.g. \(y^2 - x^2 = 1\) is also a hyperbola, \((x + 1)^2 + 2(y - 1)^2 = 1\) is an ellipse centered at \((-1, 1)\).

**Example.** \(9x^2 - 4y^2 - 72x + 8y + 176 = 0\). Complete the squares:

\[
4(y^2 - 2y) - 9(x^2 - 8x) = 176 \iff 4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144 = 36 \iff
\]

\[
4(y-1)^2 - 9(x-4)^2 = 36 \iff \frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1.
\]

This is a hyperbola; \(x\) is changed to \(y - 1\) and \(y\) is changed to \(x - 4\). The foci are \((4, 1 \pm \sqrt{13})\).

**Cylinders.** A cylinder is a surface with equation lacking one of the variables \(x, y, z\). It is parallel to the axis of this redundant variable. Say, \(y = x^2\) is a parabolic vertical cylinder above and below the parabola \(y = x^2\), \(z = 0\); \(y = z^2\) is a parabolic horizontal cylinder based on the parabola \(y = z^2\), \(x = 0\).

**Quadric surfaces.** These are the three-dimensional analogues of conic sections. They are given by quadratic equations. Their **traces** are sections by the planes parallel to coordinate planes.

**Example.** Let \(x^2 + y^2 - z^2 = -1\). Horizontal traces: \(z = k = \text{const}\), so \(x^2 + y^2 = k^2 - 1\). If \(-1 < k < 1\), this is an empty set. If \(k = \pm 1\), this is one point: \(x = y = 0\). If \(|k| > 1\), this is the circle with radius \(\sqrt{k^2 - 1}\).

Vertical traces: e.g. \(x = k\). Then \(y^2 - z^2 = -1 - k^2 \iff z^2 - y^2 = k^2 + 1\), a hyperbola. The same is for \(y = k\).

We shall study six types of quadric surfaces:
1. **Ellipsoid.** \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \). All traces are ellipses. If \( a = b = c \), this is a sphere.

2. **Cone.** \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \). Horizontal traces are ellipses, vertical traces are hyperbolas.

3. **Elliptic Paraboloid.** \( \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \). Horizontal traces are ellipses, vertical traces are parabolas.

4. **Hyperbolic Paraboloid.** \( \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \). Horizontal traces are hyperbolas, vertical traces are parabolas.

5. **Hyperboloid of One Sheet.** \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \). Horizontal traces are ellipses, vertical traces are hyperbolas. The axis of symmetry correspond to the variable with negative coefficient.

6. **Hyperboloid of Two Sheets.** \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \). Horizontal traces are ellipses, vertical traces are hyperbolas. The axis of symmetry correspond to the variable with negative coefficient.

We can also change the roles of \( x, y, z \); change \( x \) to \(-x\), shift coordinates, much like with conic sections.

**Example.** Classify the surfaces: 1. \( 4x^2 - 4x - y^2 + 2z^2 + 4z = 0 \); 2. \( x^2 + 2z^2 - 6x - y + 10 = 0 \).

1. \( 4(x^2 - x + 1/4) - y^2 + 2(z^2 + 2z + 1) = 3 \iff 4(x - 1/2)^2 + 2(z + 1)^2 - y^2 = 3 \). Dividing by 3, we get:

\[
\frac{(x - 1/2)^2}{3/4} + \frac{(z + 1)^2}{3/2} - \frac{y^2}{3} = 1.
\]

This is a hyperboloid of one sheet, centered at \((1/2, 0, -1)\), with the axis parallel to the y-axis.

2. \( x^2 - 6x + 9 - y + 2z^2 = -1 \iff (x - 3)^2 - y + 2z^2 = -1 \iff y - 1 = (x - 3)^2 + 2z^2 \). This is an elliptic paraboloid centered at \((3, 1, 0)\), with the axis parallel to the y-axis.

**Example.** Find the surface consisting of all points \( P(x, y, z) \) such that the distance from \( P \) to the \( x \)-axis is twice the distance from \( P \) to the \( yz \)-plane. The former is \( \sqrt{y^2 + z^2} \), and the latter is \( |x| \), so \( \sqrt{y^2 + z^2} = 2|x| \iff y^2 + z^2 = 4x^2 \). This is a cone.
7. Parametric Curves

Sometimes it is impossible to represent the curves on the $xy$-coordinate plane as a graph of a function $y = f(x)$. There is a more general method: parametric equations $x = f(t)$, $y = g(t)$. The variable $t$ is called a parameter. In many applications (but not necessarily), it denotes time. Such curves are called parametric curves.

Sometimes we do not put any restrictions on it, so $t$ changes from $-\infty$ to $+\infty$. Sometimes we impose conditions: $a \leq t \leq b$. In this case, the curve has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

Example. The curve $x = t^2 - 2t$, $y = t + 1$ can be sketched by plugging in different values of $t$ and calculating $x, y$.

$t = -1 \Rightarrow x = 3, y = 0$
$t = 0 \Rightarrow x = 0, y = 1$
$t = 1 \Rightarrow x = -1, y = 2$
$t = 2 \Rightarrow x = 0, y = 3$
$t = 3 \Rightarrow x = 3, y = 4$

How to convert it into Cartesian equation? Express $t$ in terms of $y$: $y = t - 1$, $x = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3$. Thus, this is a parabola.

Example. What curve is represented by $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$? $x^2 + y^2 = 1$, so this is a unit circle. $t$ is the angle, and this circle is traversed once counterclockwise (just plug in several points, e.g. $t = 0 \Rightarrow (x, y) = (1, 0)$; $t = \pi/2 \Rightarrow (x, y) = (0, 1)$), starting from the point $(1, 0)$.

Example. The curve $x = \sin 2t$, $y = \cos 2t$, $0 \leq t \leq 2\pi$: $x^2 + y^2 = 1$, so this is again a unit circle. $(0, 1)$ is both initial and terminal point (just plug in $t = 0$ and $t = 2\pi$), but note that when $t = \pi$, $(x, y)$ is also $(0, 1)$. So the unit circle is traversed twice clockwise, starting from $(0, 1)$. Why is it traversed clockwise? Because when $0 < t < \pi/4$, we have: $x = \sin 2t > 0$ and the point goes to the right from $(0, 1)$.

So we need to distinguish between a curve, which is a set of points, and a parametric curve, in which the points are traced in a certain way.

Example. Find parametric equations for the circle with center $(x_0, y_0)$ and radius $r$. If we multiply the equations of a unit circle by $r$, we get: $x = r \cos t$, $y = r \sin t$, $0 \leq t \leq 2\pi$. This is a circle with radius $r$ centered at the origin. Then just add $x_0$ and $y_0$ to obtain the necessary equations:

$x = x_0 + r \cos t$, $y = y_0 + r \sin t$, $0 \leq t \leq 2\pi$.

Example. What curve has parametric equations $x = \sin t$, $y = \sin^2 t$? $y = x^2$, but $-1 \leq x \leq 1$. So this is a piece of parabola, from $(-1, 1)$ to $(1, 1)$.

The same can be done in three dimensions: $\mathbf{r}(t) = < f(t), g(t), h(t) >$, $a \leq t \leq b$. Or, in other words, $x = f(t)$, $y = g(t)$, $z = h(t)$. We already encountered with such equations: parametric equations of lines. This was just the partial case of what we are doing now, when $f, g, h$ are linear functions of $t$.

Example. A helix $x = \cos t$, $y = \sin t$, $z = t$. Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve lies on the circular cylinder $x^2 + y^2 = 1$. The point $(\cos t, \sin t, t)$ lies directly above the point $(x, y, 0)$, which moves counterclockwise around the unit circle in the $xy$-plane. Since $z = t$, the curve spirals upward around the cylinder.

Example. A curve of intersection of $x^2 + 2y^2 = 1$ and $y + z = 2$: $x^2 + (\sqrt{2}y)^2 = 1$, so

$x = \cos t$, $\sqrt{2}y = \sin t$, $\Rightarrow y = \frac{1}{\sqrt{2}} \sin t$, $\Rightarrow z = 2 - y = 2 - \frac{1}{\sqrt{2}} \sin t$. 

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Example. [Midterm 1, Perkins, Winter 2009, 4] Find a vector function $r(t)$ that represents the curve of intersection of the surfaces $4x^2 + (z - 1)^2 = 9$ and $y = 3x^2$.

Solution. We use the same method as in Problem 1. For this problem, rewrite the first equation as

$$\left(\frac{2x}{3}\right)^2 + \left(\frac{z-1}{3}\right)^2 = 1.$$ 

Take $p = 2x/3$, $q = (z-1)/3$. There exists $t$ such that $2x/3 = \cos t$, $(z-1)/3 = \sin t$, or, after simple transformations,

$$x = \frac{3}{2} \cos t, \ z = 1 + 3 \sin t.$$ 

Finally,

$$y = 3x^2 = 3\left(\frac{3}{2}\right)^2 \cos^2 t = \frac{27}{4} \cos^2 t.$$ 

Thus,

$$r(t) = <\frac{3}{2} \cos t, \frac{27}{4} \cos^2 t, 1 + 3 \sin t>.$$ 

Example. Two particles move according to the vector functions

$$r_1(t) = <t, t, t^2> \quad \text{and} \quad r_2(s) = <0, s - 1, s^2 - s>.$$ 

Do their trajectories intersect? Do they collide?

Solution. If their trajectories intersect at some point $(x_0, y_0, z_0)$, then there exist certain values of parameters $t$ and $s$ such that

$$t = x_0, \ t = y_0, \ t^2 = z_0 \quad \text{and} \quad 0 = x_0, \ s - 1 = y_0, \ s^2 - s = z_0.$$ 

Therefore,

$$t = 0, \ \ t = s - 1, \ t^2 = s^2 - s.$$ 

From the second equation, we find $s = 1$. Note that $t = 0$ and $s = 1$ indeed satisfy the third equation $t^2 = s^2 - s$, so there is a point of intersection corresponding to the values $t = 0$ and $s = 1$. This point is $(0, 0, 0)$. The first particle comes to this point at time $t = 0$, and the second comes there at time $s = 1$. So they do not come there simultaneously, and therefore they do not collide. This is the difference between passing a crossroads safely and a car accident!
8. Calculus with Parametric Curves

Derivatives and tangent lines. For a vector function \( \mathbf{r}(t) \), its derivative at \( t_0 \) is defined as

\[
\frac{d\mathbf{r}}{dt}(t_0) = \mathbf{r}'(t_0) = \lim_{t \to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0} = \langle f'(t_0), g'(t_0), h'(t_0) \rangle.
\]

We can take the second derivative, etc. If \( \mathbf{r}(t) \) is the position of a particle at time \( t \), then \( \mathbf{r}'(t) \) is its velocity and \( \mathbf{r}''(t) \) is its acceleration at time \( t \). Its speed is \( |\mathbf{r}'(t)| \).

The rules for differentiation are the same, and the product rule \((fg)' = f'g + fg'\) holds for both dot and cross product of vector functions. (This can be verified by writing everything componentwise.)

If \( P = (f(t_0), g(t_0), h(t_0)) \) and \( Q = (f(t), g(t), h(t)) \), then \( \overrightarrow{PQ} = \langle f(t) - f(t_0), g(t) - g(t_0), h(t) - h(t_0) \rangle = \mathbf{r}(t) - \mathbf{r}(t_0) \) is a secant vector, \( (\mathbf{r}(t) - \mathbf{r}(t_0))/(t - t_0) \) is also a secant vector, and it tends to a tangent vector \( \mathbf{r}'(t_0) \) as \( t \to t_0 \). A tangent line at the point \( P \) is the line with directional vector \( \mathbf{r}'(t_0) \) which passes through \( P \). It has vector equation \( \mathbf{r} = \mathbf{r}(t_0) + \mathbf{r}'(t_0)t \), and parametric equations

\[
x = f(t_0) + f'(t_0)t, \quad y = g(t_0) + g'(t_0)t, \quad z = h(t_0) + h'(t_0)t.
\]

Example. For \( x = 2\cos t, \ y = t^2, \ z = t \), we have: \( \mathbf{r}'(t) = \langle -2\sin t, \ 2t, \ 1 > \mathbf{r}''(t) = \langle -2\cos t, \ 2, \ 0 \rangle \). At the point \( (2, 0, 0) \) we have \( t = 0 \), so \( f'(0) = 0, \ g'(0) = 0, \ h(0) = 1 \), so the tangent line is \( x = 2t, \ y = 0, \ z = 1 \).

Example. Suppose \( \mathbf{r}(t) \) is a vector function such that \( |\mathbf{r}(t)| = 1 \). Then \( \mathbf{r}'(t) \perp \mathbf{r}(t) \). Indeed, \( \mathbf{r}(t) \cdot \mathbf{r}(t) = 1 \cdot 1 = 1 \) for all \( t \); taking derivative and using the product rule, we get:

\[
\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \Rightarrow 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \Rightarrow \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0,
\]

thus \( \mathbf{r}'(t) \) is orthogonal to \( \mathbf{r}(t) \).

Example. [Final Exam, Sarantsev, Summer 2012, 2] Find the angle between the curves

\[
x = t, \ y = t, \ z = t^2 \quad \text{and} \quad x = 0, \ y = s, \ z = s^2 - s.
\]

Solution. First, let us find their point of intersection. Solve the system of equation:

\[
t = 0, \quad t = s, \quad t^2 = s^2 - s.
\]

Obviously, the unique solution is \( t = s = 0 \). Let \( \mathbf{r}_1(t) = \langle t, t, t^2 \rangle, \ \mathbf{r}_2(s) = \langle 0, s, s^2 - s \rangle \). Then \( \mathbf{r}_1'(t) = \langle 1, 1, 2t \rangle, \ \mathbf{r}_2'(s) = \langle 0, 1, 2s - 1 \rangle \). Therefore, \( \mathbf{a} = \mathbf{r}_1'(0) = \langle 1, 1, 0 \rangle \) and \( \mathbf{b} = \mathbf{r}_2'(0) = \langle 0, 1, -1 \rangle \). The angle \( \theta \) between the curves is the angle between these two vectors. Therefore,

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \pi/3.
\]

Integrals. For a vector function \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \) on \([a, b]\), we have:

\[
\int_a^b \mathbf{r}(t) dt = i \int_a^b f(t) dt + j \int_a^b g(t) dt + k \int_a^b h(t) dt.
\]

The same is true for indefinite integrals (antiderivatives). Fundamental Theorem of Calculus:

\[
\int_a^b \mathbf{r}'(t) dt = \mathbf{r}(b) - \mathbf{r}(a).
\]
**Example.** If $\mathbf{r} = 2 \cos t \mathbf{i} + \sin t \mathbf{j} + 2t \mathbf{k}$, then
\[
\int_0^{\pi/2} \mathbf{r}(t)\,dt = \mathbf{i} \int_0^{\pi/2} 2 \cos t\,dt + \mathbf{j} \int_0^{\pi/2} \sin t\,dt + \mathbf{k} \int_0^{\pi/2} 2t\,dt = 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}.
\]

**Arc length.** Here, $\mathbf{r}(t)$ is a vector function, and $v = |\mathbf{r}'|\,$ is the speed. The length of $\mathbf{r}(t)$, $\alpha \leq t \leq \beta$ (=the distance covered at time $[\alpha, \beta]$), is given by $\int_{\alpha}^{\beta} v(t)\,dt$. Indeed, split $[\alpha, \beta]$ into small subintervals $\alpha = t_0 < t_1 < \ldots < t_n = \beta$. During the time $[t_{k-1}, t_k]$, the speed is approximately $v(t_k)$, so the distance covered is $(t_k - t_{k-1})v(t_k)$. Sum up these distances and get:
\[
\sum_{k=1}^{n} v(t_k)(t_k - t_{k-1}) \approx \int_{\alpha}^{\beta} v(t)\,dt.
\]

**Example.** The length of $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq \pi$:
\[
x' = -\sin t, \ y' = \cos t, \ z' = 1, \ v(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}.
\]
Thus, the length is $\int_0^{\pi} \sqrt{2}dt = \sqrt{2}\pi$.

**Reparametrization.** Two vector functions $x = t, y = 3t$ and $x = s^3, y = 3s^3$ give the same curve. In general, for any increasing function $t = \phi(s)$ $\mathbf{R}(s) = \mathbf{r}(\phi(s))$ gives the same curve (probably its part). And $\mathbf{R}'(s) = \mathbf{r}'(t)(dt/ds)$, so the new speed $V = |\mathbf{R}'| = |\mathbf{r}'|(dt/ds) = v(dt/ds)$. We move along the same path, but with different speed. A natural reparametrization is
\[
s(t) = \int_0^t v(u)\,du \Rightarrow \frac{ds}{dt} = v \Rightarrow \frac{dt}{ds} = \frac{1}{v} \Rightarrow V = \frac{1}{v} = 1.
\]
So we move along the path with unit speed.

**Example.** For the curve above,
\[
s = \int_0^t \sqrt{2}du = \sqrt{2}t \Rightarrow x = \cos \frac{s}{\sqrt{2}}, \ y = \sin \frac{s}{\sqrt{2}}, \ z = \frac{s}{\sqrt{2}}.
\]
9. Velocity, Acceleration and Speed

The unit tangent vector is \( T = \frac{r'(t)}{|r'(t)|} = \frac{1}{v}r'(t) \). The acceleration \( a = r'' \) has two components. The first one is a tangential component, which is parallel to \( T \) (i.e. it is parallel to the tangent line; this justifies the name "tangential") and is equal to

\[
a_t := \frac{1}{v}r'' = \frac{r''}{|T|^2}T = (r'' \cdot T)T.
\]

The second one is a normal component, which is orthogonal to \( T \) and is equal to

\[
a_n := T \times (r'' \times T) = r''(T \cdot T) - T(r'' \cdot T) = r'' - a_t.
\]

"Normal" is another word for "orthogonal"; this justifies the name "normal component". Note that

\[
|a_n| = |T||r''\times T|\sin(\pi/2) = |r''\times T| = \frac{|r'' \times r'|}{v}.
\]

Denote by \( N = T'/|T'| \) the unit vector in the direction of \( T' \). Let us calculate it. After calculations, we get:

\[
T' = \frac{a_n}{v}.
\]

The meaning of \( T' \) and \( a_n \) is to measure the change of the direction of velocity, i.e. how curved is this curve, how far is it from a straight line. The two vectors \( T \) and \( N \) are orthogonal. Indeed, \( T'|a_n, \) and \( a_n \perp T \). So \( T' \perp T \), and \( N \perp T \).

The curve near the given point \( r(t) \) moves near the plane parallel to \( T \) and \( N \) and passing through \( r(t) \). This is called the osculating plane ("approximating plane"). A normal vector for the osculating plane is \( B = T \times N \), a binormal vector. A normal plane is a plane that passes through the point \( r(t) \) and is orthogonal to \( T \); it is "orthogonal to the curve at the point \( r(t) \)."

The vectors \( T, N, B \) are mutually orthogonal and have length 1. \((|B| = 1 \cdot 1 \cdot \sin(\pi/2) = 1.\)

Note that

\[
N(t) \neq \frac{r''(t)}{|r''(t)|}!!
\]

The two measures of curvature above, \( T' \) and \( a_n \), are not perfect, since after reparametrization it can change. However, a nice measure of curvature should depend only on the form of this curve, not on the speed with which we traverse it.

E.g. \( T'| \) changed by \( dt/ds \). If we divide it by \( v \) (which also changes by this factor), the ratio will be invariant under parametrization, and so \( k(t) = |T'|/v \) is called the curvature of this curve at point \( r(t) \). Let us find it:

\[
|T'| = \frac{1}{v} |a_n| = \frac{1}{v^2} |r' \times r''| \Rightarrow k(t) = \frac{|r' \times r''|}{v^3}.
\]

Example. For the curve above, \( v = \sqrt{2} \), so

\[
T(t) = \frac{r'(t)}{v} = \langle \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \rangle \Rightarrow T'(t) = \langle -\frac{1}{\sqrt{2}} \cos t, -\frac{1}{\sqrt{2}} \sin t, 0 \rangle,
\]

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The unit vector in the direction of this vector (i.e. \( \mathbf{N} \)) is \( < \cos t, -\sin t, 0 > \). Finally, 
\[
\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t/\sqrt{2} & \cos t/\sqrt{2} & 1/\sqrt{2} \\ -\cos t & -\sin t & 0 \end{vmatrix} = < \frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} > .
\]

The curvature is \( k(t) = |\mathbf{T}'|/v = 1/2 \), and \( \mathbf{a} = < -\cos t, -\sin t, 0 > \perp \mathbf{r}' \), so \( \mathbf{a}_n = \mathbf{a}, \mathbf{a}_r = \mathbf{0} \). The normal plane at \( t = 0 \), i.e. at the point \( (\cos 0, \sin 0, 0) = (1, 0, 0) \), has normal vector \( \mathbf{T}(0) = < 0, 1/\sqrt{2}, 1/\sqrt{2} > \), i.e. normal vector \( 0, 1, 1 > \), so its equation is \( 0(x-1) + 1(y-0) + 1(z-0) = 0 \Leftrightarrow y + z = 0 \). The osculating plane at \( t = 0 \) has normal vector \( \mathbf{B}(0) = < 0, -1/\sqrt{2}, 1/\sqrt{2} > \), i.e. normal vector \( 0, -1, 1 > \), so its equation is \( 0(x-1) - 1(y-0) + 1(z-0) = 0 \Leftrightarrow -y + z = 0 \).

**Example.** [Midterm 1, Sarantsev, Summer 2012, 4] Find the TNB basis of the curve 
\[
\mathbf{r} = < t, t^2/2, t >
\]
at \( t = 0 \).

**Solution.** \( \mathbf{r}'(t) = < 1, t, 1 >, \mathbf{r}'(0) = < 1, 0, 1 >. \) \( |\mathbf{r}'(t)| = \sqrt{t^2 + t^2 + 1^2} = \sqrt{2 + t^2} \), so 
\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = < \frac{1}{\sqrt{2}} , \frac{t}{\sqrt{t^2 + 2}}, \frac{1}{\sqrt{t^2 + 2}} > .
\]

At \( t = 0 \), we have:
\[
\mathbf{T}(0) = < \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} > = \frac{1}{\sqrt{2}} < 1, 0, 1 > .
\]

Moreover, 
\[
\left( \frac{1}{t^2 + 2} \right)' = (t^2 + 2)^{-1/2}' = -\frac{1}{2} (t^2 + 2)^{-3/2} (t^2)' = -\frac{t}{\sqrt{t^2 + 2}^3},
\]
\[
\left( \frac{t}{t^2 + 2} \right)' = \frac{t' \sqrt{t^2 + 2} - t(\sqrt{t^2 + 2})'}{t^2 + 2} = \frac{2}{t^2 + 2} - \frac{t}{\sqrt{t^2 + 2}^3} = \frac{2}{t^2 + 2}.
\]

Therefore, 
\[
\mathbf{T}'(t) = < \frac{t}{\sqrt{t^2 + 2}^3}, -\frac{2}{\sqrt{t^2 + 2}^3}, -\frac{t}{\sqrt{t^2 + 2}^3} > .
\]

At \( t = 0 \), we have:
\[
\mathbf{T}'(0) = < 0, \frac{2}{\sqrt{2}^3}, 0 > = \frac{1}{\sqrt{2}} < 0, 1, 0 > .
\]

The unit vector in the direction of this vector (i.e. \( \mathbf{N}(0) \)) is \( < 0, 1, 0 > \). Finally, 
\[
\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \frac{1}{\sqrt{2}} < 1, 0, 1 > \times < 0, 1, 0 > = \frac{1}{\sqrt{2}} < -1, 0, 1 > .
\]
10. Polar Coordinates

There are some useful coordinates on \( \mathbb{R}^2 \) other than Cartesian: polar coordinates. Let \( O \) be the origin and let \( P = (x, y) \) be an arbitrary point not coinciding with the origin. Suppose \( \theta \) is the angle between the \( x \)-axis (i.e. \( 1 = \langle 1, 0 \rangle \)) and \( OP \), but it can go only counterclockwise. Say, if \( P = (0, -1) \), then this angle is \( 3\pi/2 \), not \( \pi/2 \). Note that \( \theta \) can be outside \([0, \pi]\) and even outside \([0, 2\pi]\). If we add or subtract \( 2\pi \), we get essentially the same angle. Then let \( r = |OP| \). This pair \((r, \theta)\) is called polar coordinates of the point \( P \).

Note that
\[
x = r \cos \theta, \quad y = r \sin \theta; \quad r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.
\]

Only one pair of Cartesian coordinates correspond to any point other than the origin. (They differ by adding \( 2\pi \) to \( \theta \).) What about the origin? It has \( r = 0 \) and \( \theta \) is undefined. Yes, this is cumbersome, but sometimes this is really useful.

\[\begin{align*}
\theta &= 0 \Rightarrow x > 0, \quad y = 0, \quad \text{and} \quad P \text{ lies on the positive } x\text{-half-axis}, \\
\theta &\in (0, \pi/2) \Rightarrow x > 0, \quad y > 0, \quad \text{and} \quad P \text{ lies in the first quadrant.} \\
\theta &= \pi/2 \Rightarrow x = 0, \quad y > 0, \quad \text{and} \quad P \text{ lies on the positive } y\text{-half-axis, etc.} \\
\theta &\in (\pi/2, \pi) \Rightarrow x < 0, \quad y > 0, \quad \text{and} \quad P \text{ lies in the second quadrant.} \\
\theta &= \pi \Rightarrow x < 0, \quad y = 0, \quad \text{and} \quad P \text{ lies on the negative } x\text{-half-axis.} \\
\theta &\in (\pi, 3\pi/2) \Rightarrow x < 0, \quad y < 0, \quad \text{and} \quad P \text{ lies in the third quadrant.} \\
\theta &= 3\pi/2 \Rightarrow x = 0, \quad y < 0, \quad \text{and} \quad P \text{ lies on the negative } y\text{-axis.} \\
\theta &\in (3\pi/2, 2\pi) \Rightarrow x > 0, \quad y < 0, \quad \text{and} \quad P \text{ lies in the fourth quadrant.}
\end{align*}\]

**Example.** \( x = 1, \quad y = -1 \) \( \Rightarrow r = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \) \( \cos \theta = x/r = 1/\sqrt{2}, \) \( \sin \theta = y/r = -1/\sqrt{2}, \) and the pair of sine and cosine uniquely determines the angle (up to addition of \( 2\pi \)): \( \theta = -\pi/4. \)

\[r = 3, \quad \theta = 4\pi/3 \Rightarrow x = r \cos \theta = 3 \cdot (-1/2) = -3/2, \quad y = r \sin \theta = 3 \cdot (-\sqrt{3}/2) = -3\sqrt{3}/2.\]

It is easier to convert from polar to Cartesian coordinates, because in the opposite direction we need to determine the angle by its sine and cosine, this may be a bit tricky.

**Curves in polar coordinates.** Some easy examples: \( r = 2 \) is a circle of radius 2 centered at the origin; \( \theta = \pi/4 \) is the ray starting from zero in the first quadrant which bisects the right angle.

If they have the form \( r = f(\theta) \) and can be rewritten in Cartesian coordinates as \( x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta \); these are parametric equations. Then we can find tangent lines, etc.

**Example.** \( r = 2 \cos \theta \Rightarrow x = 2 \cos^2 \theta = \cos 2\theta + 1, \quad y = 2 \cos \theta \sin \theta = \sin 2\theta, \) so \( (x - 1)^2 + y^2 = \cos^2 2\theta + \sin^2 2\theta = 1. \) This is a circle with radius 1 centered at \((1, 0)\). The tangent line to this curve at \( \theta = \pi/4 \): the corresponding point on the curve is \( x = 1, \quad y = 1, \) and

\[x'(\theta) = -2 \sin 2\theta, \quad x'(\pi/4) = -2, \quad y'(\theta) = 2 \cos 2\theta, \quad y'(\pi/4) = 0,\]

so this tangent line is given by parametric equations: \( x = 1 - 2t, \quad y = 1 + 0t = 1, \) i.e. \( x = s, \quad y = 1, \) i.e. \( y = 1. \) (It is horizontal.)

**Example.** Sketch \( r = |\cos 2\theta| \). Start from \( \theta = 0 \Rightarrow r = 1. \) As \( \theta \) goes to \( \pi/4, \) \( r \) decreases to 0. When \( \theta = \pi/4, \) \( r = 0, \) and then it starts increasing again to 1 when \( \theta = \pi/2. \) Then it decreases to zero until \( \theta = 3\pi/4, \) etc. so there are four loops; this is called a four-leaved rose.
11. Partial Derivatives

**Functions of two variables.** Consider a function of two variables: \( f(x, y) \). For example, 
\( x + y, \sqrt{x^2 + y^2}, \sin(x/y) \). Examples from life: \( x \)-latitude, \( y \)-longitude, \( f(x, y) \) - height; \( x \)-temperature, \( y \)-humidity, \( f(x, y) \)-temperature-humidity index. The domain of \( f \), \( \text{dom} \ f \), is the set of all \((x, y)\) such that \( f(x, y) \) makes sense. E.g. for \( f(x) = \sin(x/y) \) \( \text{dom} \ f = \{ y \neq 0 \} \).

We can represent functions of one variable \( f(x) \) by their graphs: \( y = f(x) \), which are curves on the \( xy \)-plane \( \mathbb{R}^2 \). We can represent functions of two variables \( f(x, y) \) also by their graphs \( z = f(x, y) \), which are surfaces in \( \mathbb{R}^3 \). Or, if we wish to remain in two dimensions, we can draw contour lines \( f(x, y) = c \) for different levels \( c \): \( c = 0, 1, -1, 1/2, \) etc. Use whatever is easier.

**Example.** 1. \( f(x, y) = x + y \). The graph \( z = x + y \) is a plane in \( \mathbb{R}^3 \), and the contour lines are \( x + y = 0, x + y = 1, \ldots \), which are parallel lines on \( \mathbb{R}^2 \).

2. \( f(x, y) = x^2 + y^2 \). The graph \( z = x^2 + y^2 \) is an elliptic paraboloid, and the contour lines \( x^2 + y^2 = c \):
   - for \( c < 0 \): empty sets;
   - for \( c = 0 \): the single point (the origin);
   - for \( c > 0 \): the circle centered at the origin with radius \( \sqrt{c} \).

**Partial derivatives.** Recall: for a function \( f(x) \) of one variable its derivative at \( x_0 \) is
\[
 f'(x_0) = \frac{df}{dx}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.
\]
It is the rate of change of \( f \) at \( x_0 \). Similarly, for \( f(x, y) \), its partial derivative with respect to \( x \) at the point \((x_0, y_0)\) is
\[
 f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}.
\]
We fix \( y = y_0 \) and treat \( f \) as a function of the single variable \( x \): \( g(x) = f(x, y_0) \), and take \( g'(x_0) = f_x(x_0, y_0) \). This is the rate of change of \( f \) at \((x_0, y_0)\) in the direction of \( x \). The partial derivative with respect to \( y \) at \((x_0, y_0)\) is
\[
 f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.
\]
We fix \( x = x_0 \), so \( f \) becomes a function of \( y \): \( h(y) = f(x_0, y) \), and \( h'(y_0) = f_y(x_0, y_0) \). This is the rate of change of \( f \) in the direction of \( y \).

**Example.** 1. \( f(x, y) = 1 \Rightarrow f_x = f_y = 0 \).
2. \( f(x, y) = x + 2y^2 \Rightarrow f_x = 1, f_y = 4y \).
3. \( f(x, y) = x^2y^3 \Rightarrow f_x = 2xy^3, f_y = 3x^2y^2 \).
4. \( f(x, y) = \sin(x/y) \Rightarrow f_x = (1/y) \cos(x/y), f_y = -(x/y^2) \cos(x/y) \).

**Second-order derivatives.** Recall that for \( f(x) \), \( f'' = (f')' \): the second-order derivative is the derivative of the derivative. A function \( f(x, y) \) has 4 second-order partial derivatives: \( f_{xx} = (f_x)_x, f_{xy} = (f_x)_y, f_{yx} = (f_y)_x, f_{yy} = (f_y)_y \).

**Example.** 1. \( f(x, y) = x + 2y^2 \Rightarrow f_{xx} = 0, f_{xy} = 0, f_{yx} = 0, f_{yy} = 4 \).
2. \( f(x, y) = x^2y^3 \Rightarrow f_{xx} = 2y^3, f_{xy} = 6xy^2, f_{yx} = 6xy^2, f_{yy} = 6x^2y \).

**Clairaut’s Theorem.** If \( f_{xy} \) and \( f_{yx} \) are continuous, then they are equal.
In the sequel, these partial derivatives will always be equal.

**Linear approximation and tangent planes.** For \( f(x) \), we have: if \( x \approx x_0 \), then \( f(x) \approx f(x_0) + f'(x_0)(x - x_0) \), and \( y = f(x_0) + f'(x_0)(x - x_0) \) is the tangent line at \((x_0, f(x_0))\). Consider a function \( f(x, y) \); suppose \((x, y) \approx (x_0, y_0)\). Then for \( g(t) = f(t, y_0) \) we have:

\[
f(x, y_0) - f(x_0, y_0) = g(x) - g(x_0) \approx g'(x_0)(x - x_0) = f_x(x_0, y_0)(x - x_0).
\]

And for \( h(t) = f(x, t) \) we have:

\[
f(x, y) - f(x, y_0) = h(y) - h(y_0) \approx h'(y_0)(y - y_0) = f_y(x, y_0)(y - y_0) \approx f_y(x_0, y_0)(y - y_0),
\]

because \( x \approx x_0 \) and \( f_y(x_0, y) \approx f_y(x_0, y_0) \). Sum these equalities:

\[
f(x, y) - f(x_0, y_0) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

This is a **linear approximation for \( f \) near \((x_0, y_0)\)**. And the plane

\[
z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
\]

is called the **tangent plane to the graph of \( f \) at the point \((x_0, y_0, f(x_0, y_0))\)**.

**Example.** If the ball falls for time \( t \) with zero initial speed, the distance is \( h = gt^2/2 \), where \( g = 9.8m/c^2 \). Suppose we know \( t = 5 \pm 0.1 \), and \( g = 10 \pm 0.2m/c^2 \). So the error in each variable is 2%. What is the error in \( h \)? We have: \( t_0 = 5, \ g_0 = 10, \ |t - t_0| \leq 0.1, \ |g - g_0| \leq 0.2 \). Therefore, \( h(t_0, g_0) = 10 \cdot 5^2/2 = 125 \), and the error in \( h \) is

\[
h(t, g) - h(t_0, g_0) \approx h_t(t_0, g_0)(t - t_0) + h_g(t_0, g_0)(g - g_0).
\]

But \( h_t = gt \), so \( h_t(t_0, g_0) = 50; \ h_g = t^2/2 \), so \( h_g(t_0, g_0) = 25/2 \). So the right-hand side is dominated by \( 50 \cdot 0.1 + (25/2) \cdot 0.2 = 7.5 \). This is 6% from \( h(t_0, g_0) = 125 \). The error has accumulated, and this is even not simply the sum of errors \((2\% + 2\%)\).
12. Critical Points

Definitions. A function $f(x)$ has **global minimum** at $x_0$ if $f(x_0) \leq f(x)$ for all $x \in \text{dom } f$, i.e. if $f(x)$, whenever it is defined, is greater than or equal to $f(x_0)$. A **global maximum** is defined in the same way. Global maximum or minimum is called **global extremum**. Sometimes we will just omit the world "global".

A function $f(x)$ has **local minimum** at $x_0$ if for all points $x \approx x_0$ such that $x \in \text{dom } f$ we have: $f(x) \geq f(x_0)$. In other words, there exists a neighbourhood $U = (x_0 - \varepsilon, x_0 + \varepsilon)$ of $x_0$ such that for all $x \in U$, $x \in \text{dom } f$ we have: $f(x) \geq f(x_0)$. A **local maximum** is defined similarly. Again, local maximum and minimum are united under the term **local extremum**.

Any global extremum is a local extremum, but a local extremum may not be global. Example: $f(x) = x^3 - 3x$ has local minimum at $x = 1$, but it does not have a global minimum, since it tends to $-\infty$ as $x \to -\infty$.

The same definitions apply for $f(x,y)$, except that the ball of radius $\varepsilon$ centered at $x_0$ plays the role of $U$. The interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ was just such ball in one dimension.

**Critical points and local extrema.** Recall: if $f(x)$ has local extremum at $x_0$, then $f'(x_0) = 0$. The same is true for $f(x)$: if this function has local extremum (say, minimum) at $(x_0, y_0)$, then $g(x) = f(x, y_0)$ also has local minimum at $x = x_0$, so $g'(x_0) = 0$, i.e. $f_x(x_0, y_0) = 0$. Similarly, $f_y(x_0, y_0) = 0$.

Any solution to the system of equations $f_x(x, y) = 0$, $f_y(x, y) = 0$ is called a **critical point of** $f$. So any point of local extremum is a critical point. To find local extrema if $f$, we first need to solve this system of two equations. But the converse is not true, since a critical point may not be a point of local extremum!

**Example.** $f(x, y) = x^2 - y^2 \Rightarrow f_x = 2x$, $f_y = -2y$. The system $2x = 0$, $-2y = 0$ has a solution $(x, y) = (0, 0)$, the origin. But this is neither a local maximum nor a local minimum point. Indeed, for any $x \neq 0$ we have: $f(x, 0) = x^2 > 0 = f(0, 0)$. And points $(x, 0)$, $x \neq 0$, lie arbitrarily close to the origin (i.e. we can find such point in any $U$ from above). So the origin is not a local maximum point. It is neither a local minimum point: $f(0, y) = -y^2 < 0 = f(0, 0)$ for $y \neq 0$, and points $(0, y)$, $y \neq 0$, are arbitrarily close to the origin.

A critical point which is neither a local maximum nor a local minimum points is called a **saddle point**. For a saddle point $(x_0, y_0)$, in its any arbitrarily small neighbourhood $U$ there exist points $(x_1, y_1)$ and $(x_2, y_2)$ such that $f(x_0, y_0) > f(x_1, y_1)$ and $f(x_0, y_0) < f(x_2, y_2)$.

We need some tools to classify critical points.

**The Second Derivative Test.** Recall: if $f'(x_0) = 0$ and $f''(x_0) > 0$ or $f''(x_0) < 0$, then $f$ has local minimum/maximum at the point $x_0$. Similarly, let $(x_0, y_0)$ be a critical point of $f(x,y)$. Denote

$$
A := f_{xx}(x_0, y_0), \quad B := f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0), \quad C := f_{yy}(x_0, y_0),
$$

$$
D := \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix} = AC - B^2.
$$

If $A > 0$ and $D > 0$, $(x_0, y_0)$ is a point of local minimum.

If $A < 0$ and $D > 0$, $(x_0, y_0)$ is a point of local maximum.

If $D < 0$, $(x_0, y_0)$ is a saddle point.

**Example.** 1. $f(x, y) = x^2 - xy + y^2$. Then $f_x = 2x - y$, $f_y = x - 2y$, so $2x - y = 0 \Rightarrow y = 2x$, $x - 2 \cdot 2x = 0$, $-3x = 0$, $x = 0, y = 0$. So the origin is the only critical point. Then $f_{xx} = 2$, $f_{xy} = f_{yx} = -1$, $f_{yy} = 2$ (not just at this critical point, but everywhere), so $A = C = 2$, $B = -1$, and $D = 2 \cdot 2 - 1^2 = 3$, and the origin is a local minimum point.
2. \( f(x, y) = x^2 - 3xy + y^2 \). Then \( f_x = 2x - 3y, \ f_y = 2y - 3x, \) so \( 2x - 3y = 0 \Rightarrow x = 3y/2, \ 2y = 3x = 9y/2, \) so \( y = 0, \ x = 0. \) Again, the origin is the only critical point. Similarly, \( A = C = 2, \ B = -3, \) and \( D = -5, \) so this is a saddle point.

3. \( f(x, y) = x^4 + y^4. \) Then \( f_x = 4x^3 = 0 \Rightarrow x = 0, \ f_y = 4y^3 = 0 \Rightarrow y = 0, \) so again the origin is the only critical point, and \( f_{xx} = 12x^2 \Rightarrow A = f_{xx}(0,0) = 0, \ f_{xy} = 0 \Rightarrow B = 0, \ f_{yy} = 12y^2 \Rightarrow C = 0, \ D = AC - B^2 = 0, \) and the test fails! But this functions is obviously non-negative everywhere, and is zero at the origin, so the origin is the point of a global (therefore, a local) minimum. This function does not have any local maxima.

If we have more than two variables: \( x, y, z, \) then express \( z \) in terms of \( x \) and \( y \) (or vice versa) and reduce it to a problem with \( x \) and \( y. \)

**Example.** Minimize \( C := x^2 + y^2 + z^2 \) for \( x, y, z \geq 0, \ x + y + z = 1. \) This is a measure of concentration of capital in a market with three companies, each holding the proportion \( x, y, z \) of capital, respectively. Say, if \( x = 1, \ y = z = 0, \) then \( C = 1, \) the market is concentrated in the hands of the first owner. And if \( x = 1/2, \ y = z = 1/4, \) then \( C = 3/8, \) and the market is much less concentrated.

Here, \( z = 1 - x - y, \) and we must minimize

\[
f(x, y) = x^2 + y^2 + (1 - x - y)^2 = 2x^2 + 2y^2 + 2xy - 2x - 2y + 1.
\]

Find critical points:

\[
f_x = 4x + 2y - 2, \quad f_y = 2x + 4y - 2,
\]

and we have the system:

\[
4x + 2y - 2 = 0, \quad 2x + 4y - 2 = 0.
\]

Cancel out 2:

\[
2x + y = 1, \quad 2y + x = 1 \Rightarrow y = 1 - 2x,
\]

therefore, plugging it into the second equation, we get:

\[
2(1 - 2x) + x = 1 \Rightarrow 2 - 3x = 1 \Rightarrow x = 1/3 \Rightarrow y = 1 - 2/3 = 1/3.
\]

So \((1/3, 1/3)\) is the only critical point. Since \( A = 4, B = 2, C = 4, D = 4 \cdot 4 - 2^2 = 12, \) this is a local minimum point. (In fact, a global minimum point.)
13. Optimization on a Domain

Suppose we have a closed bounded domain \( D \subseteq \mathbb{R}^2 \). Closed means it contains its boundary \( \partial D \). Say, \( D = \{ x^2 + y^2 < 1 \} \) is not closed, since it does not contain the unit circle \( \partial D = \{ x^2 + y^2 = 1 \} \), while \( D = \{ x^2 + y^2 \leq 1 \} \) is closed. Bounded means it "does not go to infinity", i.e. contained in some rectangle.

For a function \( f(x, y) \), find its maximum and minimum on \( D \): \( M := \max_{(x, y) \in D} f(x, y) \), \( m := \min_{(x, y) \in D} f(x, y) \). Do not confuse it with local extrema! Consider two examples: \( f(x, y) = xy \) on

\[
D_1 := \{ x, y \geq 0, x + y \leq 1 \}, \quad D_2 := \{ x^2 + y^2 \leq 1 \}.
\]

**Step 1.** Use common sense. For the first domain, \( x, y \geq 0 \) and \( f \geq 0 \), but \( f(0, 0) = 0 \) and \( (0, 0) \in D_1 \). That’s why \( m = 0 \).

**Step 2.** If \( f(x_0, y_0) = M \) or \( m \) and \( (x_0, y_0) \) is strictly inside \( D \) (i.e. not on the boundary), then this is a critical point of \( f \). So let us find all critical points of \( f \) that lie strictly inside \( D \). Those critical points that lie outside \( D \) or on the boundary \( \partial D \) do not count. For the example, \( f_x = y, \ f_y = x \), so the only solution of the system \( f_x = f_y = 0 \) is the origin. It lies on \( \partial D_1 \), so it does not count for \( D_1 \), but it lies strictly inside \( D_2 \).

Note: You may perform the Second Derivative Test for these points, but you don’t have to. Indeed, if it shows that this is a critical point, you may recycle it; but if it shows this is a local maximum or minimum, this will not give you any useful information. You still have to compare it with the maximum/minimum on the boundary.

**Step 3.** Deal with the boundary \( \partial D \) separately: this is a one-dimensional curve, so the restriction of \( f \) onto \( \partial D \) can be represented as a function \( g(t) \) of one variable. We know how to find its maximum and minimum points.

For the example, \( D = D_1 \), we have: \( \partial D_1 \) it a triangle with the three sides: \( \Gamma_1 := \{ 0 \leq x \leq 1, \ y = 0 \}, \ \Gamma_2 := \{ 0 \leq y \leq 1, \ x = 0 \}, \ \Gamma_3 := \{ 0 \leq x \leq 1, \ y = 1 - x \} \). On \( \Gamma_1 \) and \( \Gamma_2 \), we have: \( f = 0 \), and on \( \Gamma_3 \), \( f = x(1 - x), \ 0 \leq x \leq 1 \). The function \( g(x) = x(1 - x) = x - x^2 \) has maximum 1/4 (at \( x = 1/2 \), \( y = 1 - x = 1/2 \)). Indeed, \( g'(x) = 1 - 2x = 0 \Rightarrow x = 1/2 \), and \( g'' = -2 < 0 \), so apply the Second Derivative Test for single-variable functions. We are not interested in the minimum, because this part is already solved in Step 1.

For \( D = D_2 \), we have: \( \partial D_2 \) is the unit circle, parametrizable by \( x = \cos t, \ y = \sin t \), \( 0 \leq t \leq 2\pi \). So \( g(t) = f(\cos t, \sin t) = \sin t \cos t = \sin(2t)/2 \) is the restriction of \( f \) onto \( \partial D_2 \). Its maxima are at \( 2t = \pi/2, 5\pi/2 \Rightarrow t = \pi/4, 5\pi/4 \), where \( g(t) = 1/2 \). Its minima are at \( 2t = 3\pi/2, 7\pi/2 \Rightarrow t = 3\pi/4, 7\pi/4 \), where \( g(t) = -1/2 \).

**Step 4.** We have found several candidates for points of maximum and minimum: \( (x_1, y_1), \ldots, (x_N, y_N) \), just plug them in \( f \) and find the maximal and minimal values.

For \( D = D_1 \) in this example, the candidates for the maximum are points on \( \Gamma_1, \ \Gamma_2 \), where \( f = 0 \), and \( (1/2, 1/2) \), where \( f = g(1/2) = 1/4 \). Clearly, \( M = 1/4 \). We know from Step 1 that \( m = 0 \).

For \( D = D_2 \), the candidates:

- the origin, where \( f = 0; \)
- \( t = \pi/4 \Rightarrow (x, y) = (\cos t, \sin t) = (1/2, 1/\sqrt{2}) \) and \( t = 5\pi/4 \Rightarrow (x, y) = (\cos t, \sin t) = (-1/\sqrt{2}, -1/\sqrt{2}) \), where \( f = g(t) = 1/2; \)
- \( t = 3\pi/4 \Rightarrow (x, y) = (\cos t, \sin t) = (-1/\sqrt{2}, 1/\sqrt{2}) \) and \( t = 7\pi/4 \Rightarrow (x, y) = (\cos t, \sin t) = (1/\sqrt{2}, -1/\sqrt{2}) \), where \( f = g(t) = -1/2 \).

Clearly, \( M = 1/2, \ m = -1/2 \).
Answer: 
\[
\max_{(x,y) \in D_1} f(x,y) = \frac{1}{4}, \quad \min_{(x,y) \in D_1} f(x,y) = 0.
\]
\[
\max_{(x,y) \in D_2} f(x,y) = \frac{1}{2}, \quad \min_{(x,y) \in D_2} f(x,y) = -\frac{1}{2}.
\]

**Example.** A package in the shape of a rectangular box can be mailed by USPS if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108in. Find the dimensions of the package with the largest volume that can be mailed.

Let \( x \) be the length, \( y \) - width, \( z \) - height. The girth is \( 2y + 2z \), and so \( x + 2y + 2z = a := 108 \text{in} \). The volume \( V = xyz \) can be expressed as a function of two variables: \( f(y, z) = yz(a - 2y - 2z) = ayz - 2y^2z - 2yz^2 \). Let us find its critical points.

\[
f_y = az - 4yz - 2z^2 = 0, \quad f_z = ay - 2y^2 - 4yz = 0,
\]
so we have (cancelling out \( y, z \)):

\[
4y + 2z = a, \quad 4z + 2y = a.
\]

Multiplying the second equation by 2 and subtracting the first equation, we get:

\[
2(4z + 2y) - (4y + 2z) = 2a - a \Rightarrow 6z = a \Rightarrow z = \frac{a}{6}.
\]

Therefore, 
\[
y = \frac{1}{2}(a - 4z) = \frac{a}{6}, \quad x = a - y - z = \frac{2a}{3}.
\]

So the dimensions which give the maximal volume are:

\[
x = 72 \text{in}, y = z = 18 \text{in}.
\]
14. Double Integrals over Rectangles

Definition of double integrals. Recall the definition of \( I := \int_a^b f(x)dx \), the integral of \( f(x) \) over the segment \([a, b]\):

1. Split \([a, b]\) into small subintervals \([x_{k-1}, x_k], k = 1, \ldots, n\), where \( a = x_0 < x_1 < \ldots < x_n = b \). The length of \([x_{k-1}, x_k]\) is \( x_k - x_{k-1} \).
2. For each subinterval, pick up a point \( u_k \in [x_{k-1}, x_k] \).
3. Create a Riemann sum
   \[
   S := \sum_{k=1}^{n} f(u_k)(x_k - x_{k-1}).
   \]
4. Let \( \max_{k=1,\ldots,n}(x_k - x_{k-1}) \to 0 \): there are more and more subintervals, and the maximal length tends to zero. Then \( S \to I \).

Similarly, let us define \( I = \iint_R f(x, y)dA \) for a rectangle \( R = [a, b] \times [c, d] \).
1. Split \( R \) into small subrectangles \([x_{k-1}, x_k] \times [y_{l-1}, y_l], k = 1, \ldots, n, \ l = 1, \ldots, m\), where \( a = x_0 < x_1 < \ldots < x_n = b, c = y_0 < y_1 < \ldots < y_m = b \). The area of \([x_{k-1}, x_k] \times [y_{l-1}, y_l]\) is \((x_k - x_{k-1})(y_l - y_{l-1})\).
2. For each of these subrectangles, pick up a point \((u_{kl}, v_{kl}) \in [x_{k-1}, x_k] \times [y_{l-1}, y_l]\).
3. Create a Riemann sum
   \[
   S := \sum_{k=1}^{n} \sum_{l=1}^{m} f(u_{kl}, v_{kl})(x_k - x_{k-1})(y_l - y_{l-1}).
   \]
4. Let \( \max(x_k - x_{k-1}) \to 0 \) and \( \max(y_l - y_{l-1}) \to 0 \): there are more and more subrectangles, and their maximal dimensions tend to zero. Then \( S \to I \).

The notation \( dA \) means “infinitesimal piece of area”; this is \((x_k - x_{k-1})(y_l - y_{l-1})\).

Reduction to iterated integrals. Consider, for example, \( R = [0, 1] \times [0, 2] \) and \( f(x, y) = xy \).

It is tempting to calculate \( \iint_R f(x, y)dA \) as an iterated integral, i.e. two single-variable integrals, the first one by \( y \) with fixed \( x \), the second one by \( x \):
\[
\iint_R f(x, y)dA = \int_0^1 \left[ \int_0^2 xydy \right] dx = \int_0^1 \frac{1}{2}xy^2 \bigg|_{y=0}^{y=2} dx = \int_0^1 2xdx = x^2 \bigg|_{x=0}^{x=1} = 1.
\]

This way of calculating double integrals is legitimate. Indeed, choose \((u_{kl}, v_{kl}) = (x_k, y_l)\), i.e. the upper right corner of the \( kl \)-tj subrectangle. Then
\[
I := \iint_R f(x, y)dA \approx S := \sum_{k=1}^{n} \sum_{l=1}^{m} f(x_k, y_l)(y_l - y_{l-1})(x_k - x_{k-1}).
\]

The inner sum \( \sum_{l=1}^{m} f(x_k, y_l)(y_l - y_{l-1}) \) is a Riemann sum for the function \( h(y) = f(x_k, y) \), so it is approximately equal to \( \int_c^d h(y)dy = \int_c^d f(x_k, y)dy \). Therefore,
\[
S \approx \sum_{k=1}^{n} \left[ \int_a^b f(x_k, y)dy \right] (x_k - x_{k-1}).
\]

The last sum is a Riemann sum for the function \( g(x) = \int_a^b f(x, y)dy \), so it is approximately equal to the integral \( \int_a^b \left[ \int_c^d f(x, y)dy \right] dx \). This suggests that
\[
\iint_R f(x, y)dA = \int_a^b \left[ \int_c^d f(x, y)dy \right] dx.
\]
Similarly, since variables $x$ and $y$ have equal rights,

$$
\iint_R f(x, y)\,dA = \int_0^d \left[ \int_a^b f(x, y)\,dx \right]\,dy.
$$

We can evaluate this iterated integral in any order: first fix $x$ and integrate by $y$, then integrate by $x$, or fix $y$ and integrate by $x$, then integrate by $y$.

**Example.** For $R = [0, 1] \times [0, 2]$ and $f(x, y) = xy$, let us reverse the order of integration

$$
\iint_R f(x, y)\,dA = \int_0^2 \left[ \int_0^1 xy\,dx \right]\,dy = \int_0^2 \frac{1}{2} x^2 y \,|x=1\rangle\,dy = \int_0^2 \frac{y^2}{2} \,|y=2\rangle = 1.
$$

**Integrating product functions.** If $f(x, y)$ is a product function, i.e. has the form $g(x)h(y)$, then

$$
\iint_R f(x, y)\,dA = \int_0^d \left[ \int_a^b g(x)h(y)\,dx \right]\,dy = \int_0^d h(y) \left[ \int_a^b g(x)\,dx \right]\,dy,
$$

because $h(y)$ is constant when we integrate by $x$. This, in turn, equals to

$$
\int_a^d h(y)I_g\,dy = I_g \int_a^d h(y)\,dy = I_g I_h, \quad I_g := \int_a^b g(x)\,dx, \quad I_h := \int_c^d h(y)\,dy.
$$

**Example.** For $R = [0, 1] \times [0, 2]$ and $f(x, y) = xy$,

$$
\iint_R f(x, y)\,dA = \int_0^1 x\,dx \int_0^2 y\,dy = \frac{1}{2} \cdot 2 = 1.
$$

**Precision of Riemann sums approximation.** Consider the integral

$$
\iint_D f(x, y)\,dA, \quad f(x, y) = xy, \quad D = [1, 3] \times [2, 4],
$$

let us calculate its value and its Riemann sums corresponding to the partition $(x_0, x_1, x_2) = (1, 2, 3)$, $(y_0, y_1, y_2) = (2, 3, 4)$. The exact value is

$$
\int_1^3 x\,dx \int_2^4 y\,dy = \frac{x^2}{2} \bigg|_{x=1}^{x=3} \frac{y^2}{2} \bigg|_{y=2}^{y=4} = \left( \frac{9}{2} - \frac{1}{2} \right) \left( \frac{16}{2} - \frac{4}{2} \right) = 24.
$$

The area of each of the four small subrectangles is 1. Suppose we select points from each of the four small rectangles in the following way: lower left corners. then the Riemann sum, which is supposed to approximate the integral, is $f(1, 2) + f(1, 3) + f(2, 2) + f(2, 3) = 2 + 3 + 4 + 6 = 15$. Pretty bad! If we choose upper left corners, then this is $f(2, 2) + f(2, 3) + f(3, 2) + f(3, 3) = 4 + 6 + 6 + 9 = 25$. Much better! For lower right corners, this sum is $f(1, 3) + f(2, 3) + f(1, 4) + f(2, 4) = 3 + 6 + 4 + 8 = 21$. Worse! For upper right corners, this is $f(2, 3) + f(3, 3) + f(2, 4) + f(3, 4) = 6 + 9 + 8 + 12 = 35$. Very bad! Mostly, the results are not very close to 24 because the subrectangles aren’t that small! We chose only $2 \times 2$ grid!
15. Double Integrals over General Regions

Let $D$ be a bounded region in $\mathbb{R}^2$, which means it is contained inside some rectangle $R$. E.g. \{y > 0\} is not bounded, while a disc, a square, a triangle are bounded. Then the integral of $f$ over $D$ is defined as

$$\iint_D f(x, y)dA = \iint_R \bar{f}(x, y)dA,$$

where $\bar{f} = f$ on $D$ and $\bar{f} = 0$ outside $D$. This is a way to reduce integration over general regions to integration over rectangles.

If $D$ is a region of type 1, i.e. enclosed between two graphs $y = \phi_1(x)$ and $y = \phi_2(x)$: $D = \{\phi_1(x) \leq y \leq \phi_2(x), \ a \leq x \leq b\}$, then enclose $D$ into $R = [a, b] \times [c, d]$:

$$\iint_D f(x, y)dA = \iint_R \bar{f}(x, y)dA = \int_a^b \left[ \int_c^d \bar{f}(x, y)dy \right] dx = \int_a^b \left[ \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \right] dx.$$  

Indeed, for $y$ between $\phi_1(x)$ and $\phi_2(x)$ we have: $(x, y) \in D$ and $\bar{f}(x, y) = f(x, y)$. For $y$ in $[c, d]$, but outside $[\phi_1(x), \phi_2(x)]$ we have: $(x, y) \notin D$ and $\bar{f}(x, y) = 0$.

**Example.** For $D = \{0 \leq x \leq 1, \ x \leq y \leq x + 1\}$ and $f(x, y) = xy$, we have:

$$\iint_D f(x, y)dA = \int_0^1 \left[ \int_x^{x+1} xydy \right] dx = \int_0^1 \frac{xy^2}{2} \bigg|_{y=x}^{y=x+1} dx = \int_0^1 \frac{x}{2}((x+1)^2 - x^2)dx = \int_0^1 \frac{x^2}{2} + \frac{x}{2} dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right) \bigg|_{x=0}^{x=1} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$  

Similarly, if $D$ is a region of type 2, i.e. $D = \{c \leq y \leq d, \ \phi_1(y) \leq x \leq \phi_2(y)\}$, then

$$\iint_D f(x, y)dA = \int_c^d \left[ \int_{\phi_1(y)}^{\phi_2(y)} f(x, y)dx \right] dy.$$  

**Change of integration order.** If $D$ is both type 1 and 2 region, we can integrate in any order, just as we had for rectangles (because integration over general regions reduces to integration over rectangles).

**Example.** $D = \{x, y \geq 0, \ y \leq 1 - x^2\}$. Then $D$ is a type 1 region: $D = \{0 \leq x \leq 1, \ 0 \leq y \leq 1 - x^2\}$, and

$$\iint_D f(x, y)dA = \int_0^1 \left[ \int_0^{1-x^2} f(x, y)dy \right] dx.$$  

But $D$ is also a type 2 region: $D = \{0 \leq y \leq 1, \ 0 \leq x \leq \sqrt{1-y}\}$, so

$$\iint_D f(x, y)dA = \int_0^1 \left[ \int_0^{\sqrt{1-y}} f(x, y)dx \right] dy.$$  

**Example.** [Min Wu, Midterm 2, Summer 2011, 3] Evaluate the iterated integral

$$I := \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}dydx.$$  

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**Solution.** We are not able to integrate $e^{y^3}$. So we cannot compute this iterated integral directly. But this is a double integral over $D = \{0 \leq x \leq 1, \sqrt{x} \leq y \leq 1\} = \{0 \leq y \leq 1, 0 \leq x \leq y^2\}$. So

$$I = \int\int_D e^{y^3} \, dA = \int_0^1 \int_0^{y^2} e^{y^3} \, dx \, dy = \int_0^1 y^2 e^{y^3} \, dy,$$

because $e^{y^3}$ is constant with respect to $x$. Change variables $u = y^3$, $du = 3y^2 \, dy$, $0 \leq u \leq 1$, then

$$\int_0^1 y^2 e^{y^3} \, dy = \int_0^1 e^{u} \, \frac{du}{3} = \frac{1}{3}(e - 1).$$

**Geometric meaning.** Just like $\int_a^b f(x) \, dx$ for $f \geq 0$ is the area under the graph $y = f(x)$ and above the segment $[a,b]$, the double integral

$$\int\int_D f(x,y) \, dA$$

for a positive function $f$ is the volume under the surface $z = f(x,y)$ and above the domain $D$. If $f = 1$, then this integral

$$\int\int_D 1 \, dA$$

is equal to the area of $D$.

**Example.** [Final Exam, Spring 2010, 6] Find the volume of the solid that lies under the plane $3x + 2y + z = 12$ and above the rectangle $R = [0, 1] \times [2, 3]$.

**Solution.** This body is between $z = 0$ and $z = 12 - 3x - 2y$ and above the rectangle $R$. So its volume is

$$\int\int_R (12 - 3x - 2y) \, dA = \int_2^3 \int_0^1 (12 - 3x - 2y) \, dx \, dy.$$

But

$$\int_0^1 (12 - 3x - 2y) \, dx = \left(12x - \frac{3x^2}{2} - 2xy\right)\bigg|_{x=0}^{x=1} = 12 - \frac{3}{2} - 2y = \frac{21}{2} - 2y.$$

Therefore,

$$\int_2^3 \int_0^1 (12 - 3x - 2y) \, dx \, dy = \int_2^3 \left(\frac{21}{2} - 2y\right) \, dy = \left(\frac{21}{2} y - y^2\right)\bigg|_{y=2}^{y=3} = \frac{21}{2} - 5 = \frac{11}{2}.$$
16. Double Integrals in Polar Coordinates

Can we change variables in double integrals in the same way as for single-variable integrals? Yes, we can, but the general formula is quite cumbersome [Math 324]; now we shall study the most important change of variables: from Cartesian to polar coordinates: \( x = r \cos \theta, \ y = r \sin \theta \). We should rewrite \( D \) and \( f \) in polar coordinates.

Say, for \( f = \sqrt{x^2 + y^2}, \ D = \{ x^2 + y^2 \leq 1 \} \), we have: \( f = r, \ D = \{ 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1 \} \). (Do not forget: \( r \geq 0 \) always, and if there are no restrictions on \( \theta \), then set \( 0 \leq \theta \leq 2\pi \)) So, we can write

\[
\int_D \int f(x, y) \, dA = \int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta.
\]

But this is wrong! Actually, we must add multiple \( r \) plays the role of, say, \( 2x \) in the change of variable \( u = x^2, \ du = 2x \, dx \) for single-variable integrals.

Why is this? Because the small element of \( D \) has area \( r \, dr \, d\theta \), not \( dr \, d\theta \). (While in Cartesian coordinates it has area \( dx \, dy \).)

Here, the role of rectangles is played by \( \textit{polar rectangles} \): \( \{ \alpha \leq \theta \leq \beta, \ R_1 \leq r \leq R_2 \} \). And the area \( \Delta A \) of small polar rectangle

\[
\{ \theta_0 \leq \theta \leq \theta_0 + \Delta \theta, \ R \leq r \leq R + \Delta R \}
\]

is approximately \( R \Delta R \Delta \theta \), not \( \Delta R \Delta \theta \). Indeed, its area is proportional to the angle width \( \Delta \theta \), so \( \Delta A = \frac{\Delta \theta}{2\pi} S \), where \( S \) is the area of the ring enclosed between the two circles with radii \( R \) and \( R + \Delta R \) (centered at the origin). But

\[
S = \pi(R + \Delta R)^2 - \pi R^2 = 2\pi R \Delta R + \pi(\Delta R)^2 \approx 2\pi R \Delta R,
\]

because \( \Delta R \) is small and \((\Delta R)^2 \) is really small. Thus, \( \Delta A = R \Delta R \Delta \theta \).

Let us finish this example:

\[
\int_D \int f(x, y) \, dA = \int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^1 r^2 \, dr = \frac{2\pi}{3}.
\]

More generally, for \( D = \{ \alpha \leq \theta \leq \beta, \ r_1(\theta) \leq r \leq r_2(\theta) \} \)

\[
\int_D \int f(x, y) \, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.
\]

We can symbolically represent this as \( dA = dx \, dy = r \, dr \, d\theta \).

**Example.** The area of the disc with radius \( R \) is \( \pi R^2 \). The volume of the ball with radius \( R \) is \((4/3)\pi R^3\). What is the 4D volume of the 4D ball \( \{ x^2 + y^2 + z^2 + w^2 \leq R^2 \} \) of radius \( R \)?

**Solution.** It is \( cR^4 \), where \( c \) is the volume of the unit ball \( B = \{ x^2 + y^2 + z^2 + w^2 \leq 1 \} \). For any \( (x, y, z, w) \in B \), we have: \( x^2 + y^2 \leq x^2 + y^2 + z^2 + w^2 \leq 1 \). Fix any \( x, y \) such that \( x^2 + y^2 \leq 1 \). The section of \( B \) is the disc \( \{ z^2 + w^2 \leq 1 - x^2 - y^2 \} \) of radius \( \sqrt{1 - x^2 - y^2} \) centered at the origin. Its area is \( \pi(\sqrt{1 - x^2 - y^2})^2 = \pi(1 - x^2 - y^2) \). So

\[
c = \iiint_{\{x^2+y^2\leq1\}} \pi(1 - x^2 - y^2) \, dA = \pi \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta = \frac{\pi}{2} \iiint_{\{x^2+y^2\leq1\}} \frac{2\pi}{3} (r^2 \, dr \, d\theta) = \frac{\pi^2}{2}.
\]

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Thus, $c = \frac{\pi^2}{2}$. The 4D volume of the 4D ball with radius $R$ is $(\frac{\pi^2}{2})R^4$.

**Example.** [Final Exam, Winter 2009, 8] Let

$$D = \{(x, y) \in \mathbb{R}^2 \mid 4 \leq x^2 + y^2 \leq 4x, \ y \geq 0\}.$$  

(a) Draw a careful picture of $D$.
(b) Compute the area of $D$.

**Solution.**

(a) $\{4 \leq x^2 + y^2\}$ is the exterior of the circle with radius 2 centered at the origin. $\{x^2 + y^2 \leq 4\}$ is the interior of the circle centered at $(2, 0)$ with radius 2. So this is the intersection of these two domains.

(b) In polar coordinates, $x^2 + y^2 = r^2$, $x = r \cos \theta$, and let us find $D$ in polar coordinates.

$$4 \leq x^2 + y^2 \iff r \geq 2; \ x^2 + y^2 \leq 4x \iff r^2 \leq 4r \cos \theta \iff r \leq 4 \cos \theta.$$  

So the limits on $r$: 

$$2 \leq r \leq 4 \cos \theta.$$  

Also, $y = r \sin \theta \geq 0 \iff \sin \theta \geq 0$. Let us find limits on $\theta$: $2 \leq r \leq 4 \cos \theta$ $\Rightarrow$ $\cos \theta \geq \frac{1}{2}$. So we have (draw the unit circle!):

$$\cos \theta \geq 1/2, \ \sin \theta \geq 0 \iff 0 \leq \theta \leq \pi/3.$$  

Finally, 

$$D = \{0 \leq \theta \leq \pi/3, \ 2 \leq r \leq 4 \cos \theta\}.$$  

The area of this domain is

$$\int_0^{\pi/3} \int_2^{4 \cos \theta} r \ dr \ d\theta = \int_0^{\pi/3} \left[ \frac{r^2}{2} \right]_2^{4 \cos \theta} \ d\theta = \int_0^{\pi/3} \left[ \frac{1}{2}(16 \cos^2 \theta - 4) \right] d\theta.$$  

But $(16 \cos^2 \theta - 4)/2 = 8 \cos^2 \theta - 2 = 4(1 + \cos 2\theta) - 2 = 2 + 4 \cos 2\theta$. Therefore, this integral is equal to

$$(2 \sin 2\theta + 2\theta)_{\theta = \pi/3}^{\theta = 0} = 2 \sin (2\pi/3) - 2 \sin 0 + \frac{2\pi}{3} = \sqrt{3} + \frac{2\pi}{3}.$$  

**Remark.** If we had

$$D = \{4 \leq x^2 + y^2 \leq 4x\},$$

then we would have just $\cos \theta \geq 1/2$, i.e. $-\pi/3 \leq \theta \leq \pi/3$. Do not hesitate to make $\theta$ negative! Just be sure it does not overlap, i.e. the interval for $\theta$ is not wider than $2\pi$. Writing $-\pi/2 \leq \theta \leq 2\pi$ is wrong!

If you do not have any conditions on $\theta$ (as in the 4D ball example), then simply let $0 \leq \theta \leq 2\pi$. 

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17. Applications of Double Integrals

Suppose the lamina occupies a region \( D \subseteq \mathbb{R}^2 \) and its density at the point \((x, y)\) is \( \rho(x, y) \). Then the total mass is

\[
M = \int\int_D \rho(x, y) dA.
\]

This can be applied to electric charge and its density instead of mass, etc.

Suppose we have a pointmass \( m \) at a signed distance \( r \) from a certain axis or at a distance \( r \) from a point. E.g. for the \( x \)-axis \( r = y \), for the \( y \)-axis \( r = x \), and for the origin \( r = \sqrt{x^2 + y^2} \). Then \( mr \) is a moment about this axis or point, and \( mr^2 \) is a moment of inertia about this axis or point.

Example. If we have a pointmass \( m \) at \((-1, 2)\), then:
- the moment about the \( x \)-axis is \( 2m \);
- the moment about the \( y \)-axis is \(-m\);
- the moment of inertia about the \( x \)-axis is \( 4m \);
- the moment of inertia about the \( y \)-axis is \( m \);
- the moment of inertia about the origin is \( 5m \).

If we have a finite number of such pointmasses, just add up their moments to get a corresponding moment of the whole system. The moment of the system equals the sum of moments. For the lamina above, split it into small pieces of mass \( m_k = \rho(x_k, y_k) dA \) at points \((x_k, y_k)\). Then the moments about the \( x \)-axis and \( y \)-axis are approximately

\[
\sum_{k=1}^n m_k y_k \approx \int\int_D y \rho(x, y) dA =: M_x, \quad \sum_{k=1}^n m_k y_k \approx \int\int_D x \rho(x, y) dA =: M_y.
\]

Similarly, moments of inertia around the \( x \)-axis, \( y \)-axis and the origin are

\[
I_x := \int\int_D y^2 \rho(x, y) dA, \quad I_y := \int\int_D x^2 \rho(x, y) dA, \quad I_0 := \int\int_D (x^2 + y^2) \rho(x, y) dA.
\]

The center of mass of the lamina is \((\bar{x}, \bar{y})\), where

\[
\bar{x} = \frac{M_x}{M} = \frac{\int\int_D x \rho(x, y) dA}{\int\int_D \rho(x, y) dA}, \quad \bar{y} = \frac{M_y}{M} = \frac{\int\int_D y \rho(x, y) dA}{\int\int_D \rho(x, y) dA}.
\]

The moment of inertia plays much the same role in rotational motion that mass plays in linear motion. The moment of inertia of a wheel is what makes it difficult to start or stop the rotation.

The radius of gyration of a lamina about an axis is the number \( R \) such that \( MR^2 = I \), where \( I \) is the moment of inertia about the given axis. So if the mass of the lamina were concentrated at a distance \( R \) from the axis then the moment of inertia would be the same. In particular, the radii of gyration \( x_g, y_g \) around the \( y \)- and \( x \)-axes are defined by \( Mx_g^2 = I_y, \quad My_g^2 = I_x \).

Example. When \( D \) is the contiguous United States (i.e. excluding Alaska and Hawaii) and \( \rho = 1 \), the center of mass is called the geographical center. It is near Lebanon, Kansas, a bit to the south from the boundary between Kansas and Nebraska. If we include Alaska and Hawaii, it moves to north-west, to Belle Fourche, South Dakota (right at the border between Montana and Wyoming).

When \( \rho \) is the density of population, we get the center of population. For contiguous US, it is near Plato, Missouri. For the whole US, it is also there (shifted a few miles to south-west).
Example. [Midterm 2, Taggart, Spring 2011, 5] The boundary of a lamina consists of the semicircles

\[ y = \sqrt{1 - x^2} \quad \text{and} \quad y = \sqrt{25 - x^2} \]

and the portions of the x-axis that join them.

The density of the lamina at any point is inversely proportional to its distance from the origin. That is, there is a constant \( k \) such that, the density of the lamina at the point \((x, y)\) is

\[ \rho(x, y) = \frac{k}{\sqrt{x^2 + y^2}}. \]

Find the center of mass of the lamina.

(You may use the fact that, by symmetry, the center of mass is on the y-axis.)

Solution. Suppose \( m \) is the center of mass of the lamina, \((x_c, y_c)\) is its center of mass. By symmetry, it lies on the y-axis, which is equivalent to \( x_c = 0 \), and it suffices to find \( y_c \).

\[
y_c = \frac{1}{M} \iint_D y \rho(x, y) \, dx \, dy, \quad M = \iint_D \rho(x, y) \, dx \, dy, \]

where \( D \) is the region occupied by lamina. Let us convert these integrals to polar coordinates.

\[
D = \{ 1 \leq r \leq 5, \quad 0 \leq \theta \leq \pi \},
\]

because the first semicircle has radius 1, the second one has radius \( \sqrt{25} = 5 \), and \( D \) lies above the x-axis (this implies restrictions on \( \theta \)). Moreover,

\[
\rho = \frac{k}{r}, \quad dA = r \, dr \, d\theta, \quad y = r \sin \theta.
\]

Therefore,

\[
M = \int_0^\pi \int_1^5 \frac{k}{r} r \, dr \, d\theta = k \int_1^5 dr \int_0^\pi d\theta = 4k\pi,
\]

\[
\int_0^\pi \int_1^5 r \sin \theta \frac{k}{r} r \, dr \, d\theta = k \int_0^\pi \sin \theta d\theta \int_0^5 r \, dr = 24k
\]

(after calculating these integrals). Thus,

\[
y_c = \frac{24k}{4k\pi} = \frac{6}{\pi}.
\]

Example. For the homogeneous disc \( D \) centered at the origin with radius \( a \) and density \( \rho \), we have: \( D = \{ 0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi \} \) and \( dA = r \, dr \, d\theta \) in polar coordinates, \( \rho(x, y) = \rho = \text{const} \), and

\[
I_0 = \iint_D (x^2 + y^2) \rho(x, y) \, dA = \int_0^{2\pi} \int_0^a r^2 \, r \, dr \, d\theta = \int_0^a r^3 \, dr \int_0^{2\pi} d\theta = \frac{\pi \rho a^4}{2}.
\]

Since \( I_x + I_y = I_0 \) and \( I_x = I_y \) from the symmetry of the problem, we have: \( I_x = I_y = \frac{\pi \rho a^4}{4} \).

The mass is \( M = \pi \rho a^2 \). The radius of gyration \( x_g = \sqrt{I_y/M} = \sqrt{a^2/4} = a/2 \), and \( y_g = a/2 \) similarly.
18. Linear Taylor Approximation

Linear approximation error bound. For a function $f(x)$, we have: $f(x) \approx f(0) + f'(0)x = T_1(x)$ for $x \approx 0$. How exact is this approximation? We would like to find an exact inequality, an estimate for $f(x) - T_1(x)$. Let $x > 0$, then

$$f(x) - T_1(x) = f(x) - f(0) - f'(0)x = \int_0^x f'(t)dt - \int_0^x f'(0)dt = \int_0^x [f'(t) - f'(0)]dt = \int_0^x udv,$$

where $u = f'(t) - f'(0)$ and $v = t - x$. Integrate by parts:

$$\int_0^x udv = u(x)v(x) - u(0)v(0) - \int_0^x vdu = -\int_0^x (t - x)f''(t)dt = \int_0^x (x - t)f''(t)dt.$$

Indeed, $u(0) = 0$ and $v(x) = 0$, so $u(x)v(x) - u(0)v(0) = 0$. Thus,

$$f(x) = f(0) + f'(0)x + \int_0^x (x - t)f''(t)dt.$$

This integral is the magnitude of the error. We can estimate it: if $M_2 = \max_{0 \leq t \leq x} |f''(t)|$, then

$$\left| \int_0^x (x - t)f''(t)dt \right| \leq M_2 \int_0^x (x - t)dt = M_2 \int_0^x udv = \frac{M_2}{2}x^2.$$

The same is true for $x < 0$, except that $\int_0^x = -\int_x^0$ (say, $\int_0^{-1} = -\int_x^0$), and the maximum is taken over all $t$ such that $x \leq t \leq 0$.

This is called Linear Taylor Error Bound, or Tangent Line Error Bound. $T_1(x)$ is called first-order Taylor polynomial, or Tangent Line Approximation, or Linear Taylor Approximation.

Example. $f(x) = e^x$. Then $f(0) = 1$, $f'(0) = e^x$, $f'(0) = 1$, so $e^x \approx T_1(x) = 1 + x$ for $x \approx 0$. E.g. $e^{0.1} \approx 1.1$. The error is $(0.1^2/2)M_2$, where $M_2 := \max_{0 \leq t \leq 0.1} |f''(t)| = \max_{0 \leq t \leq 0.1} e^t = e^{0.1} \leq 2$. So the error is dominated by $(0.1^2/2) \cdot 2 = 0.01$, i.e. less than 1% - this is very good for physical experiments!

Establishing the given error. Consider the following type of problem: fix some small $\varepsilon$, say 0.1, 0.01, etc., and find $a > 0$ such that for all $x \in [-a, a]$ we have $|f(x) - T_1(x)| \leq \varepsilon$. We need: for all $x \in [-a, a]$

$$\frac{x^2}{2} \max_{-a \leq t \leq a} |f''(t)| \leq \varepsilon.$$

Here, the maximum is taken over $0 \leq t \leq x$ for positive $x$ and for $x \leq t \leq 0$ for negative $x$. But all such $t$ lie between $-a$ and $a$, because $x \in [-a, a]$. And $x^2 \leq a^2$. So anyway, if we establish

$$\frac{a^2}{2} \max_{-a \leq t \leq a} |f''(t)| \leq \varepsilon;$$

this would give us the required $a$.

Example. $f(x) = \sin x$, $\varepsilon = 0.01$. Then $f''(x) = -\sin x$, and $|f''(t)| = |\sin t| \leq 1$. So

$$\frac{a^2}{2} \max_{-a \leq t \leq a} |f''(t)| \leq \frac{a^2}{2},$$

and if we let $a^2/2 = 0.01 \Leftrightarrow a = \sqrt{0.02} \approx 0.141$, this is a correct answer. If we are a bit smarter, we will note that for all $t$, $|\sin t| \leq |t|$, then this maximum is less than or equal to $a$, and

$$\frac{a^2}{2} \max_{-a \leq t \leq a} |f''(t)| \leq \frac{a^3}{2}.$$
So if \( a^{3/2} = 0.01 \Leftrightarrow a = \sqrt[3]{0.01} = 0.271 \), this is also a correct answer. There are many correct answers to such problems. In fact, if some \( a \) is correct, then any smaller positive \( a \) is also correct. The larger is \( a \), the better, since the interval \([-a, a]\) becomes larger, and for more and more \( x \) we have \( |f(x) - T_1(x)| \leq \varepsilon \). You do not have to find the best answer; any valid answer is good, provided it is properly justified.

**Taylor approximation based not at zero.** If you approximate \( f(x) \) not for \( x \approx 0 \), but for \( x \approx b \), then \( f(x) \approx T_1(x) = f(b) + f'(b)(x - b) \), and the difference is

\[
f(x) - T_1(x) = \int_b^x (x - t)f''(t)dt, \ |f(x) - T_1(x)| \leq \frac{(x - b)^2}{2} M_2, \ M_2 := \max |f''(t)|,
\]

where the maximum is taken over \( t \) between \( b \) and \( x \).

**Example.** Let \( f(x) = \sqrt{x} \), \( b = 1 \). Then \( f(b) = 1, \ f'(x) = 1/(2\sqrt{x}), \ f'(b) = 1/2. \) So \( T_1(x) = 1 + (x - 1)/2. \)

For \( x = 1.1, \ f(x) = \sqrt{1.1} \approx 1.05. \)

For \( x = 1.4, \ f(x) = \sqrt{1.4} \approx 1.2. \)

For \( x = 1.01, \ f(x) = \sqrt{1.01} \approx 1.005. \)

What is the error for \( x = 1.1? \) Since \( f''(t) = -1/(4t^{3/2}) \), the function \( |f''(t)| = 1/(4t^{3/2}) \) is decreasing. So \( M_2 = \max_{1 \leq t \leq 1.1} |f''(t)| = |f''(1)| = 1/4. \) And the error bound is \( (0.1^2/2)M_2 = 0.00125. \) Less than 0.2%!
19. Quadratic Taylor Approximation

**Quadratic approximation.** Recall: for $x > 0$

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt.$$ 

Note that

$$\int_0^x (x-t)f''(t)dt - \frac{x^2}{2}f''(0) = \int_0^x (x-t)f''(t)dt - \int_0^x uf''(0)du =$$

$$\int_0^x (x-t)f''(t)dt - \int_0^x (x-t)f''(0)dt = \int_0^x (x-t)(f''(t) - f''(0))dt = \int_0^x udv,$$

where $u = f''(t) - f''(0)$ and $v = -(x-t)^2/2$, because $dv = -(x-t)d(x-t) = (x-t)dt$. Integrate by parts: note that $u(0) = 0$ and $v(x) = 0$, so $u(x)v(x) - u(0)v(0) = 0$, and

$$\int_0^x udv = u(x)v(x) - u(0)v(0) - \int_0^x vdu = \frac{1}{2} \int_0^x (x-t)^2 f''(t)dt.$$

We used $du = f''(t)dt$. Summarize:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{1}{2} \int_0^x (x-t)^2 f''(t)dt.$$ 

The quadratic polynomial $T_2(x) = f(0) + f'(0)x + f''(0)/2x^2$ is called quadratic approximation or Taylor’s second-order polynomial. The integral in the right-hand side is the error $f(x) - T_2(x)$. If $M_3 = \max_{0 \leq t \leq x} |f'''(t)|$, then this error has absolute value bounded by

$$\frac{1}{2} \int_0^x (x-t)^2 M_3 dt = \frac{M_3}{2} \int_0^x (x-t)^2 dt = \frac{M_3}{2} \int_0^x u^2 du = \frac{M_3 x^3}{6} = \frac{M_3 |x|^3}{6}.$$ 

This is called quadratic approximation error bound. The same results are true for $x < 0$, with the following corrections: $M_3$ is the maximum over all $x \leq t \leq 0$, and $\int_0^x = -\int_x^0$.

**Example.** $f(x) = e^x$, then $f(0) = 1$, $f'(x) = e^x$, $f'(0) = 1$, $f''(x) = e^x$, $f''(0) = 1$, so for $x \approx 0$

$$e^x \approx T_2(x) = 1 + x + \frac{x^2}{2}.$$ 

How exact is this approximation? Is it better than $e^x \approx T_1(x) = 1 + x$?

For $x = 0.1$, $e^{0.1} \approx T_2(0.1) = 1.105$, and the error bound is $(0.1^3/6) M_3$, $M_3 := \max_{0 \leq t \leq 0.1} |f'''(t)| = \max_{0 \leq t \leq 0.1} e^t = e^{0.1} \leq 2$, so the error is bounded by $0.1^3/3 \approx 0.00033$. This is much better than the error bound 0.01 for $f(x) - T_1(x)$.

Adding a new term $f''(0)x^2/2$ to the linear approximation, i.e. upgrading form linear to quadratic approximation, improves this approximation. It takes longer to compute it, but it is better.

**Establishing a certain precision.** If we wish to establish a certain precision, then $T_2$ allows to do this for larger interval of $x$ around zero then $T_1$.

Fix $\varepsilon > 0$, say 0.01, and find $a > 0$ such that for all $x \in [-a, a]$ we have: $|f(x) - T_2(x)| \leq \varepsilon$.

The error bound is

$$\frac{|x|^3}{6} M_3, \ M_3 := \max |f'''(t)|,$$
where this max is taken for all \( t \) between 0 and \( x \). Since \( x \in [-a, a] \), all these \( t \) also lie between \([-a, a]\), and

\[
M_3 \leq \max_{-a \leq t \leq a} |f''(t)|,
\]

and \(|x|^3 \leq a^3\). So we need to establish

\[
\frac{a^3}{6} \max_{-a \leq t \leq a} |f''(t)| \leq \varepsilon.
\]

**Example.** \( f(x) = \sin x, \varepsilon = 0.01 \). Then \( f(0) = 0, f'(x) = \cos x, f'(0) = 1, f''(x) = -\sin x, f''(0) = 0 \), so \( T_2(x) = x \). Actually, in this case \( T_2 = T_1 \): there is no quadratic term, so nothing is added to \( T_1 \) to get \( T_2 \). So the quadratic and linear approximations coincide. But the quadratic approximation error bound gives us better results than linear approximation error bound: \(|f''(t)| = | - \cos t | = | \cos t | \), so \( \max_{-a \leq t \leq a} |f''(t)| = 1 \) (the max is attained at \( t = 0 \)), and we need \( a^3/6 \leq 0.01 \). Let \( a^3/6 = 0.01 \iff a = \sqrt[3]{0.06} = 0.391 \), which is better than 0.271 for the tangent line error bound.

**Quadratic approximation based not at zero.** If we approximate \( f(x) \) for \( x \approx b \) instead of \( x \approx 0 \), then

\[
f(x) = T_2(x) + \frac{1}{2} \int_b^x (x - t)^2 f'''(t) dt, \quad T_2(x) = f(b) + f'(b)(x - b) + \frac{f''(b)}{2} (x - b)^2,
\]

and the error bound is

\[
\frac{M_3}{6} |x - b|^3, \quad M_3 := \max |f'''(t)|,
\]

where the maximum is taken over all \( t \) between \( b \) and \( x \).

**Example.** \( f(x) = x^2, b = 1 \). Then \( f(b) = 1, f'(b) = 2, f''(b) = 2 \), so \( T_2(x) = 1 + 2(x - 1) + 2(x - 1)^2/2 = 1 + 2(x - 1) + (x - 1)^2 = (1 + (x - 1))^2 = x^2 \). Actually, the quadratic approximation coincides with the function! No surprize - the function is itself quadratic. And \( T_1(x) = 1 + 2(x - 1) \), which is only approximately equal to \( x^2 \).
20. General Taylor Polynomials

Recall: after integrating by parts twice, we get:

\[ f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{1}{2} \int_0^x (x-t)^2 f'''(t)dt. \]

Integrating by parts once more, we get:

\[ f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{1}{6} \int_0^x (x-t)^3 f''''(t)dt, \]

etc. Recall that \(f^{(k)}\) is the \(k\)th order derivative of \(f\). So for any \(n\) we get:

\[ f(x) = T_n(x) + \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)dt, \quad T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots + \frac{f^{(n)}(0)}{n!} x^n. \]

This polynomial \(T_n\) is called \(n\)th Taylor polynomial, or Taylor polynomial of degree \(n\), or \(n\)th order Taylor approximation. The integral is the error.

Here, \(n! = 1 \cdot 2 \cdot 3 \cdots n\) is called the factorial of \(n\). E.g. \(1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, \) etc. We always have: \((n-1)! \cdot n! = n!\) for \(n \geq 2\). Plug in \(n = 1\): \(0! \cdot 1! = 1\), so it is reasonable to accept a convention that \(0! = 1\) (although the product of zero factors does not make sense).

Let us find an error bound. Suppose that \(x > 0\) (the case \(x < 0\) is similar) and let \(M_{n+1} = \max_{0 \leq t \leq x} |f^{(n+1)}(t)|\). Then we have:

\[ \frac{M_{n+1}}{n!} \int_0^x (x-t)^n dt = \frac{M_{n+1}}{n!} \int_0^x u^n du = \frac{M_{n+1}}{n!(n+1)} x^{n+1} = \frac{|x|^{n+1}}{(n+1)!} M_{n+1}. \]

The same bound is valid for \(x < 0\), with the following amendment: the maximum is taken over \(x \leq t \leq 0\).

**Example.** \(f(x) = e^x\). All derivatives are the same, they are equal to \(e^x\), so \(f^{(k)}(0) = 1\) for all \(k\). Therefore,

\[ T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!}. \]

For \(x = 0.1\) we have: \(M_{n+1} = \max_{0 \leq t \leq 0.1} e^t = e^{0.1} \leq 2\), so the error bound is

\[ \frac{2 \cdot 0.1^{n+1}}{(n+1)!}. \]

For \(n = 1, 2\) we already know this: it is 0.01 for \(n = 1\) and 0.00033 for \(n = 2\). For \(n = 3\), this is \(10^{-4}/12 < 10^{-5}\) - much less!

The larger \(n\) is, the better the approximation is. If you fix \(x\), then \(|f(x) - T_n(x)|\) gets smaller and smaller. Suppose you fixed some small \(\epsilon > 0\) and you want to find \(a > 0\) such that for all \(x \in [-a, a]\) you have \(|f(x) - T_n(x)| \leq \epsilon\). Then you need

\[ \frac{a^{n+1}}{(n+1)!} \max_{-a \leq t \leq a} |f^{(n+1)}(t)| \leq \epsilon. \]

**Example.** \(f(x) = \sin x, n = 4, \epsilon = 0.01\). Then \(f(0) = 0, f'(x) = \cos x, f'(0) = 1, f''(x) = -\sin x, f''(0) = 0, f'''(x) = -\cos x, f'''(0) = -1, f^{(4)}(x) = \sin x, f^{(4)}(0) = 0\). So

\[ f(x) = \sin x \approx T_4(x) = x - \frac{x^3}{6}. \]
Since \( f^{(5)}(t) = \cos t \), we need:
\[
\frac{a^5}{5!} \max_{-a \leq t \leq a} |\cos t| \leq 0.01.
\]
This max is equal to 1 (for all \( t \) we have \(|\cos t| \leq 1\), while \(|\cos 0| = 1\)). So we need \( a^5/5! \leq 0.01 \). If we let \( a^5/5! = 0.01 \Leftrightarrow a^5/120 = 0.01 \Leftrightarrow a^5 = 1.2 \Leftrightarrow a = \sqrt[5]{1.2} \geq 1 \), so \( a = 1 \) is valid. (Remember, we do not need to find the best, i.e. the largest possible \( a \).) Recall that for \( T_1 \) we had \( a = 0.271 \), and for \( T_2 \) we had \( a = 0.391 \). The more terms we take, the more precise this approximation is.

**Example.** For \( f(x) = \sin x \), find \( n \) such that \(|f(x) - T_n(x)| \leq 10^{-4}\) for all \(-1 \leq x \leq 1\). We need to find \( n \) such that for \( a = 1 \) we have:
\[
\frac{a^{n+1}}{(n+1)!} \max_{-a \leq t \leq a} |f^{(n+1)}(t)| \leq 10^{-4}.
\]
The derivative of any order of \( f \) is either \( \pm \sin t \) or \( \pm \cos t \); anyway, \(|f^{(n+1)}(t)| \leq 1\), so this max is dominated by 1. So we need: \( a^{n+1}/(n+1)! \leq 10^{-4} \). Plug in \( a = 1 \): \( 1/(n+1)! \leq 10^{-4} \Leftrightarrow (n+1)! \geq 10^4 = 10000 \). Try some \( n \): for \( n = 5 \) we have \((n+1)! = 6! = 720 < 10000\); for \( n = 6 \) we have \((n+1)! = 7! = 5040 < 10000\), and for \( n = 7 \) we have \((n+1)! = 8! = 40320 > 10000\). So \( n = 7 \) is a correct answer (of course, larger values of \( n \) are also answers). And
\[
f(x) = \sin x \approx T_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}.
\]

Finally, if we approximate \( f(x) \) for \( x \approx b \), then
\[
f(x) = T_n(x) + \frac{1}{n!} \int_b^x (x-t)^n f^{(n+1)}(t)dt,
\]
\[
T_n(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f'''(b)}{6}(x-b)^3 + \ldots + \frac{f^{(n)}(b)}{n!}(x-b)^n.
\]
And the error bound is
\[
\frac{|x-b|^{n+1}}{(n+1)!} M_{n+1}, \quad M_{n+1} := \max |f^{(n+1)}(t)|,
\]
where the maximum is taken over \( t \) between \( b \) and \( x \).
21. Series

Simple series. Consider the sum

\[ S_1 := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}. \]

Is it finite or infinite? Note that \(1/3 + 1/4 > 1/4 + 1/4 = 1/2, 1/5 + 1/6 + 1/7 + 1/8 > 1/8 + 1/8 + 1/8 + 1/8 = 1/2, 1/9 + \ldots + 1/16 > 1/16 + \ldots + 1/16 = 1/2, \ldots \) So \( S_1 > 1 + 1/2 + 1/2 + 1/2 + \ldots = \infty \), thus \( S_1 = \infty \). This is called the harmonic series. A series is just an infinite sum of real numbers. It either converges to some number, which is called the sum of the series, or it diverges. Thus, the harmonic series diverges.

Finally, consider the sum

\[ S_2 := 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}? \]

Note that

\[ \frac{1}{2^2} < \frac{1}{1 \cdot 2} = 1 - \frac{1}{2}, \quad \frac{1}{3^2} < \frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}, \quad \frac{1}{4^2} < \frac{1}{3 \cdot 4} = \frac{1}{3} - \frac{1}{4}, \ldots, \quad \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}. \]

So

\[ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots < 1 + \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots = 2. \]

So \( S_2 \leq 2 \), and \( S_2 \) is finite, i.e. this series converges. Actually, \( S_2 = \pi^2/6 \). This was proved by Euler in the 18th century. We can prove it in Math 309.

Finally, consider the sum

\[ S_3 := 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^3}. \]

This series also converges, since each term is less than or equal to the corresponding term from \( S_2 \): \( 1/n^3 \leq 1/n^2 \) for all \( n \geq 1 \). So \( S_3 \leq S_2 \). The exact value of \( S_3 \) is unknown. (Open problem!)

A series can have both positive and negative terms. Say, the series

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \]

converges, despite the harmonic series (with all pluses) diverges. This series converges because the signs alternate and \( 1/n \to 0 \). The sum of this series is actually \( \ln 2 \). We shall show this soon.

Criterion for convergence of series. By the way, if \( \sum_{n=1}^{\infty} a_n = S \) converges, then \( a_1 + \ldots + a_n \to S \) as \( n \to \infty \). Also, \( a_1 + \ldots + a_{n-1} \to S \) as \( n \to \infty \), so the difference \( a_n = (a_1 + \ldots + a_n) - (a_1 + \ldots + a_{n-1}) \to S - S = 0 \). Thus: every convergent series has terms tending to zero. If the terms do not tend to zero, the series diverges.

Note that it works only in one direction: if terms do not tend to zero, the series diverges, but if terms do tend to zero, the series may converge (as \( \sum 1/n^2 \)) or diverge (as \( \sum 1/n \)).

Geometric series.

\[ S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \Rightarrow \frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = S - 1. \]
So $S/2 = 1$, $S = 2$. Similarly, for any $x \in (-1, 1)$ we have:

$$
S = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \Rightarrow xS = x + x^2 + x^3 + \ldots = S - 1 \Rightarrow S = \frac{1}{1-x}.
$$

This is called geometric series, because its terms form a geometric progression. For $x$ outside the interval $(-1, 1)$, we have: $x^n \to 0$ (say, for $x = -2$ we have $(x^n) = (-2, 4, -8, 16, \ldots)$ and this sequence does not tend to zero). So the geometric series diverges. We say that the interval of convergence of geometric series is $(-1, 1)$.

**The ratio test.** Suppose

$$
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda.
$$

Does the series $\sum a_n$ converge? Approximately, $|a_n|$ is the geometric sequence with common ratio $\lambda$. If $\lambda > 1$, then the geometric series $\sum \lambda^n$ diverges, and this suggests that $\sum a_n$ also diverges. If $\lambda < 1$, then the geometric series $\sum \lambda^n$ converges; this suggests that $\sum a_n$ also converges, and in fact this is true.

If $\lambda = 1$, either is possible: $\sum 1/n$ diverges, $\sum 1/n^2$ converges, and as $n \to \infty$, we have:

$$
\frac{1/(n+1)}{1/n} = \frac{n}{n+1} = \frac{1}{1/n+1} \to \frac{1}{0+1} = 1, \quad \frac{1/(n+1)^2}{1/n^2} = \left( \frac{n}{n+1} \right)^2 \to 1^2 = 1.
$$

**Example.** $\sum n^2/2^n$ converges, since, as $n \to \infty$ we have:

$$
\frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \frac{1}{2} \frac{(n+1)^2}{n^2} = \frac{1}{2} \left( 1 + \frac{1}{n} \right)^2 \to \frac{1}{2} \cdot 1^2 = 1/2 < 1.
$$
22. Power Series and Taylor Series

**Power series.** This is the series of the form \( \sum_{n=0}^{\infty} c_n x^n \), where \( c_n \) are some numbers, \( x \) is a variable. Assume the following limit exists:

\[
\lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = R.
\]

Apply the Ratio Test: \( a_n = c_n x^n \), so

\[
\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x}{R} \right|.
\]

So if \( |x| < R \), this series converges, and if \( |x| > R \), it diverges. The interval \((-R, R)\) is called the **interval of convergence**, and \( R \) is the **radius of convergence**. We cannot say anything about the points \( \pm R \): the series may converge or diverge.

**Example.**
1. \( \sum x^n / n! \): \( c_n = 1/n! \), so \( |c_n/c_{n+1}| = (1/n!)/(1/(n+1)!) = (n+1)!/n! = n+1 \to \infty \), so \( R = \infty \), it converges for all real \( x \). If there are factorials in the denominator of \( c_n \), the interval of convergence is the whole real line. In fact, this series converges to \( e^x \).
2. \( \sum_{n \geq 1} (-1)^{n+1} x^n / n \): \( c_n = (-1)^{n+1}/n \), and \( |c_n/c_{n+1}| = (n+1)/n = 1+1/n \to 1 \), so \( R = 1 \). In fact, the sum is \( \ln(1 + x) \). If \( c_n \) contains only powers of \( n \) in the numerator and/or denominator (possibly with alternating signs - it does not make a difference), then \( R = 1 \).

**Taylor series.** We have Taylor approximation for \( f(x) \), \( x \approx 0 \):

\[
f(x) \approx T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{6} x^3 + \ldots + \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.
\]

The larger \( n \) is, the better this approximation is, because the error bound usually becomes smaller and smaller and tends to zero as \( n \to \infty \). If we set \( n = \infty \), in most cases we will have exact equation:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.
\]

This is called **Taylor series for \( f(x) \) centered at 0**. We can make it centered at \( b \):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!} (x - b)^n.
\]

**Standard Taylor series.** Let us calculate such series for the following four easiest functions: \( 1/(1-x) \), \( e^x \), \( \sin x \), \( \cos x \).

1. \( f(x) = 1/(1-x) \). Actually, we already know this:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.
\]

2. \( f(x) = e^x \). Then all derivatives are also \( e^x \), and their values at zero are 1. So

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\]
3. \( f(x) = \sin x \). Then
- \( f^{(0)}(0) = f(0) = \sin 0 = 0 \);
- \( f^{(1)}(0) = f'(0) = \cos 0 = 1 \);
- \( f^{(2)}(0) = f''(0) = -\sin 0 = 0 \);
- \( f^{(3)}(0) = f'''(0) = -\cos 0 = -1 \);
- \( f^{(4)}(0) = \sin 0 = 0 \);
- \( f^{(5)}(0) = \cos 0 = 1 \);
- \( f^{(6)}(0) = -\sin 0 = 0 \);
- \( f^{(7)}(0) = -\cos 0 = -1 \), etc.

Therefore, the Taylor series has the form

\[
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

We can write it in closed form. Indeed,

\[
f^{(n)}(0) = \begin{cases} 0, & n = 2k; \\ (-1)^k, & n = 2k + 1, \end{cases}
\]

so

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\]

4. \( f(x) = \cos x \). Then similarly

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.
\]

The first series (geometric series) has the interval of convergence \((-1, 1)\), and the three other series (for \(e^x\), \(\cos x\), \(\sin x\)) converge for all \(x\).
23. Operations with Taylor Series

1. Substitution.

\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots
\]

\[
\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots
\]

The interval of convergence for the second series is \(-1 < -x^2 < 1\); the right inequality is always true, and the left \((-x^2 > -1)\) is true iff \(-1 < x < 1\).

2. Addition and multiplication by a constant.

\[
2 \cos x + e^x = \sum_{n=0}^{\infty} 2\left(\frac{(-1)^n x^{2n}}{(2n)!}\right) + \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\]

Note that we were not able to write this series using one sigma sign in the form of

\[
\sum_{n=0}^{\infty} c_n x^n
\]

for some coefficients \(c_n\), because the terms \(x^n\) and \(x^{2n}\) have different powers. We could, of course, write it as

\[
\sum_{n=0}^{\infty} \left[\frac{2(-1)^n x^{2n}}{(2n)!} + \frac{x^n}{n!}\right],
\]

but this is confusing since each term contains two expressions \(x^k\) with different powers \(k\). There is nothing wrong with this. In this case, it might be easier to write the first few terms without the sigma notation:

\[
2 \cos x + e^x = 2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right) + 1 + x + \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^6}{24} + \ldots = 3 + x + \frac{3}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{8} x^4 + \ldots.
\]

The interval of convergence of the sum is the minimal of the two intervals of convergence. Say, in this case both series converge everywhere (i.e. the interval of convergence is \(\mathbb{R}\)), that’s why the resulting series also converges everywhere.

3. Multiplication or division by \(x^k\).

\[
x e^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \ldots
\]

\[
e^x - \frac{1}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = 1 + \frac{x}{2} + \frac{x^2}{6} + \ldots
\]

This was because

\[
e^x - 1 = \sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!},
\]
the first term (corresponding to \( n = 0 \)) just cancels out, and we start summation from the second term \( (n = 1) \). We can change the summation limits: let \( n = k + 1 \), then \( k = 0, 1, 2, \ldots \), and
\[
\sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}.
\]
The interval of convergence is preserved.

4. Differentiation.
\[
\frac{1}{(1-x)^2} = \left( \frac{1}{1-x} \right)' = \sum_{n=0}^{\infty} (x^n)' = \sum_{n=0}^{\infty} nx^{n-1}.
\]
The first term (corresponding to \( n = 0 \)) is just zero, so we can skip it and start summation from \( n = 1 \):
\[
\sum_{n=1}^{\infty} nx^{n-1} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \ldots
\]
Here, we substituted \( n = k + 1 \), \( k = 0, 1, \ldots \), as before. The interval of convergence is preserved.

5. Integration.
\[
\arctan x = \int_0^x \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots
\]
\[- \log(1-x) = \int_0^x \frac{dt}{1-t} = \sum_{n=0}^{\infty} \int_0^x t^n dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots
\]
And then substitute \( y = -x \) and get:
\[- \log(1+y) = \sum_{n=1}^{\infty} \frac{(-y)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n y^n}{n}.
\]
Mutliply by \(-1\):
\[
\log(1+y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} y^n}{n} = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots
\]
Plug in \( x = 1 \) and \( x = 1/\sqrt{3} \) in the arctan series and get:
\[
\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots, \quad \frac{\pi}{6} = \arctan \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} - \frac{1}{3} \left( \frac{1}{\sqrt{3}} \right)^3 + \frac{1}{5} \left( \frac{1}{\sqrt{3}} \right)^5 - \ldots
\]
Using these series, one can calculate the approximate value of \( \pi \). The second series is preferable, since it converges faster (its terms decrease to zero faster than that for the first series).
Also, plug in \( y = 1 \) for \( \log(1+y) \) series and get: \( \ln 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots \)
The interval of convergence is preserved.
Past Exam Problems

1. Analytic Geometry. Chapter 12

**Problem 1.** [Midterm 1, Milakis, Spring 2007, 1] Let \( a = \langle 1, 1, x \rangle \) and \( b = \langle 0, x, 1 \rangle \).

(a) Find \( x \in \mathbb{R} \) such that \( a \) and \( b \) are orthogonal.
(b) Find \( x \in \mathbb{R} \) such that the angle between \( a \) and \( b \) is \( \pi/4 \).

**Problem 2.** [Midterm 1, Gunnarsson, Spring 2007, 2] Consider the two vectors \( a = \langle 1, 2, 3 \rangle \) and \( b = \langle 2, 3, 4 \rangle \). Calculate the following:

(a) The cosine of the angle between \( a \) and \( b \);
(b) \( a \times b \);
(c) The area of the parallelogram with corner points \( P(0, 0, 0), Q(1, 2, 3), R(2, 3, 4), S(3, 5, 7) \).

**Problem 3.** [Midterm 1, Loveless, Winter 2007, 1bc] Let \( a = \langle 3, −1, 2 \rangle \) and \( b = 5i − 7j + 2k \).

(b) Find \( a \cdot b \).
(c) Find the angle, \( \theta \), between the vectors \( a \) and \( b \). Give your answer in radians such that \( 0 \leq \theta \leq \pi \). (Round your answer to 3 digits after the decimal point.)

**Problem 4.** [Midterm 1, Loveless, Winter 2007, 2] (a) Find all values of \( x \) so that \( a = \langle 1, x, −4 \rangle \) and \( b = \langle x, 3, 5 \rangle \) are orthogonal.
(b) Find the unit vector that is orthogonal to both \( a = \langle 1, 4, 5 \rangle \) and \( b = \langle −1, 3, 0 \rangle \).

**Problem 5.** [Midterm 1, Conroy, Winter 2006, 5] Suppose the vector \( \langle x, 3, 2 \rangle \) is orthogonal to the vector \( \langle 2, 3, x \rangle \). Find \( x \).

**Problem 6.** [Midterm 1, Milakis, Winter 2009, 2] (a) Check if the planes \( x + 3y + z = 2 \) and \( 2x + y − z = −1 \) are parallel. If not, find a parametric equation of the line of intersection of the two planes.
(b) Find, correct to the nearest degree, the angle between these two planes.

**Problem 7.** [Midterm 1, Milakis, Winter 2009, 3] (a) Determine whether the lines \( r_1(t) = \langle 2, −1, 0 \rangle + t \langle −1, 1, 1 \rangle \) and \( r_2(s) = \langle 1, 3, 0 \rangle + s \langle −2, −1, 3 \rangle \) are parallel, skew or intersecting. If they intersect, find the point of intersection.
(b) Find (if exists) an equation of the plane that contains these lines.

**Problem 8.** [Midterm 1, Conroy, Spring 2007, 3] Find the equation of the plane containing the line of intersection of the two planes

\[ x + y + z + 5 = 0 \text{ and } 3x + 2y − z + 2 = 0 \]

and the point \((1, 2, 1)\).

**Problem 9.** [Midterm 1, Gunnarsson, Spring 2007, 5] Let \( v = \langle 1, 3, −1 \rangle \) and \( r_0 = \langle 1, 1, 1 \rangle \) and consider the line given by \( r = r_0 + tv \), in vector form. Also, consider the plane given by \( x + 2y + 2z + 2 = 0 \).
(a) Show that the line and the plane are not parallel.
(b) Find the point on the line at distance 3 from the plane.

**Problem 10.** [Final Exam, Spring 2007, 3] True - False Quiz.
1. \( \mathbf{v} \times \mathbf{v} = \mathbf{0} \) only if \( \mathbf{v} = \mathbf{0} \).
2. \( \mathbf{v} \cdot \mathbf{v} = 0 \) only if \( \mathbf{v} = \mathbf{0} \).
3. \( \mathbf{u} \times (\mathbf{v} \cdot \mathbf{w}) \) makes sense.
4. \( \mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) \) makes sense.
5. \( \mathbf{v} \cdot \mathbf{w} \) is a vector that is perpendicular to both \( \mathbf{v} \) and \( \mathbf{w} \).
6. \( \mathbf{v} \times \mathbf{w} \) is a vector that is perpendicular to both \( \mathbf{v} \) and \( \mathbf{w} \).
7. For all vectors \( \mathbf{v} \) and \( \mathbf{w} \) we have \( (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} \).
8. For all vectors \( \mathbf{v} \) and \( \mathbf{w} \) we have \( (\mathbf{v} - \mathbf{w}) \times (\mathbf{v} + \mathbf{w}) = \mathbf{v} \times \mathbf{v} - \mathbf{w} \times \mathbf{w} \).

**Problem 11.** [Midterm 1, Milakis, Spring 2007, 2] Let \( A \) be the point \((1, 2, 3)\) and \( O \) the origin \((0, 0, 0)\). Consider the points \( P(x, y, z) \) such that

\[
\mathbf{AP} \cdot \mathbf{OP} - \mathbf{OA} \cdot \mathbf{OP} = 2 - |\mathbf{OA}|^2.
\]

Show that the set of all such points is a sphere, and find its center and radius.

**Problem 12.** [Midterm 1, Conroy, Spring 2007, 5] Let \( S \) be the surface defined as the set of points \( p \) (in three-dimensional space) such that the distance from \( p \) to the plane \( y = 5 \) equals the distance from \( p \) to the line \( y = 1, z = 2 \).

(a) Find an equation for \( S \).
(b) Find the equation of the trace of \( S \) in the plane \( z = 6 \). Describe the trace (i.e. what kind of curve is it?).

**Problem 13.** [Midterm 1, Pevtsova, Winter 2007, 5] Find the angle between the two main diagonals of a unit cube.

(A unit cube is a cube with the vertices \((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\), the main diagonals are the diagonals connecting the vertex \((0, 0, 0)\) with the vertex \((1, 1, 1)\), and the vertex \((1, 0, 0)\) with the vertex \((0, 1, 1)\)).

**Problem 14.** [Midterm 1, Goebel, Spring 2006, 5] Is the triangle with vertices \((2, 0, 0), (4, 3, 5), (0, 1, 3)\) a right triangle? Clearly justify your answer.

**Problem 15.** [Final Exam, Spring 2010, 4] (a) Find the equation of the plane containing the line

\[
x = 1 + 3t, \ y = 2 + 2t, \ z = 3 + t
\]

and the point \((0, 2, 5)\).
(b) Write the equation of the line of intersection of the two planes defined by \( 2x - z = 0 \) and \( x + y + z = 1 \).

**Problem 16.** [Midterm 1, Bekyel, Autumn 2007, 1] For the questions below use the points \( P(2, 1, 5), Q(1, 3, 4) \) and \( R(3, 0, 6) \).
(a) Find a vector orthogonal (perpendicular) to the plane through the points \( P, Q \) and \( R \).
(b) Find the area of the triangle \( PQR \).
(c) Determine if the point \( T(0, 3, 3) \) is on the same plane as \( P, Q \) and \( R \).
Problem 17. [Midterm 1, Pevtsova, Autumn 2006, 4] Let $A = (3, 0, 0)$, $B = (0, 4, 0)$, and $C = (0, 0, 1)$.
(a) Find the area of the triangle $ABC$.

Hint. The following identity may be useful: $3^2 + 4^2 + 12^2 = 13^2$.
(b) Let $CH$ be the height of the triangle from the vertex $C$ to the base $AB$. Find the coordinates of the point $H$.

Problem 18. [Midterm 2, Milakis, Spring 2007, 3] (a) Find an equation of the plane through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

(b) Compute the distance from the point $P(2, 0, 0)$ to the plane described in part (a).

Problem 19. [Midterm 1, Conroy, Spring 2006, 3] Find a vector that is orthogonal to the vector $(11, 3, -5)$ and has length 7.

Problem 20. [Midterm 1, Pevtsova, Winter 2007, 4] Check whether the points $(1, 2, 3)$, $(-2, 5, 7)$, and $(-5, 8, 11)$ lie on the same line.

Problem 21. [Midterm 1, Conroy, Autumn 2010, 1] Find the angle between the vectors $<3, 4, -1>$ and $<5, 2, 8>$.

Problem 22. [Midterm 1, Conroy, Spring 2010, 4] Let $S$ be the surface in 3D consisting of all points which are twice as far from the $z$-axis as they are from the $x$-axis.
(a) Give an example of a point on this surface, other than the origin.
(b) Give an equation for this surface.
(c) Describe this surface (if it is a quadric surface, categorizing it (i.e., ellipsoid, elliptic paraboloid, etc.) is sufficient).

2. Differential Geometry. Chapters 10 and 13

Problem 23. [Midterm 2, Bekyel, Spring 2008, 1] Sketch the region bounded by
\[ z = 3x^2 + y^2 \text{ and } z = 6 - 3x^2 - y^2 \]
and find parametric equations for the intersection of the two surfaces.

Problem 24. [Midterm 1, Perkins, Winter 2009, 4] Find a vector function $\mathbf{r}(t)$ that represents the curve of intersection of the surfaces $4x^2 + (z - 1)^2 = 9$ and $y = 3x^2$.

Problem 25. [Midterm 1, Bekyel, Spring 2009, 2] Find the angle of intersection of the two curves $\mathbf{r}_1(t) = <t^3, 2t^2 + 1, 2t + 3>$ and $\mathbf{r}_2(s) = <s - 4, s - 3, s - 1>$.

Problem 26. [Midterm 1, Milakis, Winter 2009, 6] Find the exact coordinates of the lowest point on the curve in $\mathbb{R}^2$ given by the parametric equations $x = 2\cos(t) + \sin(t)$, $y = \sin(t) - \cos(t)$.

Problem 27. [Midterm 1, Bekyel, Spring 2008, 3] Sketch the graph of the curve
\[ x = e^t \cos t, \ y = e^t \sin t, \ 0 \leq t \leq 2\pi \]
marking the $x$ and $y$ intercepts and find its length.
Problem 28. [Final Exam, Spring 2007, 5] What is the maximal curvature of the curve $y = \ln \cos x$?

Problem 29. [Midterm 2, Milakis, Spring 2009, 4] A curve is given by the equation $r = 2(1 - \cos \theta)$ in polar coordinates.
(a) Sketch the curve.
(b) Find all the points on the curve where the tangent line is horizontal.

Problem 30. [Midterm 2, Bekyel, Autumn 2007, 3] Consider the curve $r(t) = \langle t^2, \cos(t^3), \sin(t^3) \rangle$.
(a) Find the length of the curve from $t = 0$ to $t = 2\pi$.
(b) Reparametrize the curve with respect to arc length measured from the point $t = 0$.

Problem 31. [Midterm 2, Pevtsova, Winter 2007, 1] Consider the curve given by the equation $r = 4 \cos \theta + \sin \theta$ in polar coordinates.
(a) Find the Cartesian equation of the curve. Sketch the curve.
(b) Find the equation of the tangent line to the curve at the point $\theta = \pi/4$.

Problem 32. [Midterm 2, Milakis, Winter 2009, 2] Reparametrize the curve $\langle 2t^2 + 1, 4t, 1 \rangle$ with respect to arc length measured from point $(1, 0, 1)$ in the direction of increasing $t$. Express the reparametrization in its simplest form.

Problem 33. [Midterm 2, Pevtsova, Winter 2007, 2] Let $r(t) = \langle t, t^2, t^3 \rangle$. Find the equation of the normal plane:
(a) at $t = 1$;
(b) at $r(t) = \langle -1, 1, -1 \rangle$;
(c) find the parametric equations of the line of intersection of these two normal planes.

Problem 34. [Midterm 2, Pevtsova, Winter 2007, 4] (a) Find the velocity and position vectors $v(t)$, $r(t)$ of a particle if $a(t) = \langle 2, \cos t, \sin t \rangle$, and at the moment $t = 0$ we have $v(0) = \langle 0, 0, -1 \rangle$, $r(0) = \langle 1, 1, 0 \rangle$.
(b) Find the curvature of $r(t)$ at $t = 1$.

Problem 35. [Final Exam, Spring 2010, 2a] The location of a particle is given by the vector function $r(t) = \langle 3t - 6, 2t^3 - 5t, -t^2 + 11 \rangle$.
Find the speed of the particle at the instant when it passes through the $yz$-plane.

Problem 36. [Midterm 2, Milakis, Spring 2007, 5] Let $r(t) = \langle t^2, 2t, t \rangle$.
(a) Find the unit tangent vector $T$, the unit normal vector $N$ and the binormal vector $B$ at the point where $T$ is parallel to the plane $x + y + z = 0$.
(b) Find the curvature of $r(t)$ at the point identified in part (a).
Problem 37. [Final Exam, Spring 2007, 6a] An object is moving so that its position at time \( t \) (where \( t > 0 \)) is given by the vector function \( \mathbf{r}(t) = \langle t, 1/t, t^2 \rangle \). Find all values of \( t \) at which its acceleration vector is orthogonal to its velocity vector.

Problem 38. [Midterm 2, Bekyel, Spring 2008, 2] For the curve given by
\[
\mathbf{r}(t) = \langle 4 \sin t, 3t, 4 \cos t \rangle
\]
(a) Find the unit tangent vector \( \mathbf{T}(t) \).
(b) Find the unit normal vector \( \mathbf{N}(t) \).
(c) Find parametric equations for the tangent line to the curve at the point \((2, \pi/2, 2\sqrt{3})\).
(d) Find the equation of the normal plane to the curve at the point \((2, \pi/2, 2\sqrt{3})\).

Problem 39. [Midterm 2, Milakis, Winter 2009, 1] (a) Sketch the curve \( r = 1 + \cos \theta \) for \( \theta \in [0, 3\pi] \).
(b) Find the points on the previous curve where the tangent line is horizontal or vertical.

Problem 40. [Midterm 2, Pevtsova, Winter 2007, 5] Show that if a particle moves with a constant speed, then the velocity and acceleration vectors are orthogonal.

Problem 41. [Midterm 2, Conroy, Spring 2007, 3] Suppose a particle is moving in 3-dimensional space so that its position vector is
\[
\mathbf{r}(t) = \langle t, t^2, 1/t \rangle.
\]
(a) Find the tangential component of the particle’s acceleration vector at time \( t = 1 \).
(b) Find all values of \( t \) at which the particle’s velocity vector is orthogonal to the particle’s acceleration vector.

Problem 42. [Midterm 2, Conroy, Spring 2007, 4] Consider the curve in the \( xy \)-plane defined by the position vector function
\[
\mathbf{r}(t) = \langle t^2 - 3t, t^2 + 2t \rangle.
\]
Find the \( t \)-value of the point of maximum curvature on this curve.

Problem 43. [Midterm 2, Milakis, Spring 2007, 4] A curve is given by the equation \( r = 2(1 - \cos \theta) \) in polar coordinates.
(a) Sketch the curve.
(b) Find all the points on the curve where the tangent line is horizontal.

3. Partial Derivatives. Chapter 14

Problem 44. [Midterm 2, Gunnarsson, Spring 2007, 5] Let
\[
z = f(x, y) = \sqrt{e^x/y}.
\]
(a) Find the domain of \( f \).
(b) Calculate \( f_x, f_y, \) and \( f_{xy} \).
(c) Sketch the level curves for \( z = 1 \) and \( z = 2 \).
Problem 45. [Midterm 2, Pevtsova, Winter 2007, 3] Consider the surface given by the equation
\[ f(x, y) = x^2 y + y^3 + x. \]

(a) Find the tangent plane to the surface at the point \((-2, 1, 3)\).
(b) Find all second partial derivatives of \(f\).

Problem 46. [Midterm 2, Arms, Autumn 2006, 4] For the function \(f(x, y) = x^2 \sin(\pi y)\):
(a) Compute \(f_x\), \(f_y\), and \(f_{xy}\).
(b) Find the equation of the tangent plane to the graph of \(f\) at the point where \((x, y) = (3, 1)\).
(c) Find the equations of the line through \((3, 1, f(3, 1))\) and perpendicular to the tangent plane in part (b).

Problem 47. [Midterm 2, Perkins, Winter 2009, 4] You wish to build a rectangular box with no top with volume 6 ft\(^3\). The material for the bottom is metal and costs $3.00 a square foot. The sides are wooden and cost $2.00 a square foot. Calculate the dimensions of the box with minimum cost. Use the Second Derivative test to verify that your answer is indeed a minimum.

Problem 48. [Midterm 2, Milakis, Winter 2009, 3] Find the tangent plane to the surface given by the graph of
\[ f(x, y) = \sqrt{28 - 2x^2 - y^2} \]
at \((2, 2)\). Use the linear approximation to estimate \(f(1.95, 2.01)\).

Problem 49. [Midterm 2, Milakis, Winter 2009, 4] Find (if any) the absolute maximum and minimum values of
\[ f(x, y) = 3xy^2 \]
in \(D = \{(x, y) : x \geq 0, \ y \geq 0, \ x^2 + y^2 \leq 9\}\).

Problem 50. [Final Exam, Spring 2007, 8] Let
\[ f(x, y) = \frac{2x^2 + y^2}{\ln(2x - y)}. \]

(a) Find and sketch the domain of \(f\).
(b) Consider the surface \(z = f(x, y)\). Find the equation of the tangent plane to the surface at a point \((x_0, y_0, z_0)\) with \(x_0 = y_0 = e\).
(c) Using the linear approximation at \((e, e)\) estimate \(f(3, 3)\).

Problem 51. [Midterm 2, Perkins, Winter 2009, 2] Find the equation of the tangent plane of the function
\[ F(x, y) = \frac{3y - 2}{5x + 7} \]
at the point \((1, 1)\).

Problem 52. [Final Exam, Spring 2008, 9] Find three positive numbers \(x, y, \) and \(z\) whose sum is 100 and for which the product
\[ xy^2z^3 \]
is maximal.
Problem 53. [Final Exam, Autumn 2007, 9] Find the local maximum and minimum values and the saddle points of the function
\[ f(x, y) = x^3 - 12x - 6y + y^2 + 1. \]

Problem 54. [Final Exam, Spring 2007, 9] You wish to build a large swimming pool in the shape of a parallelepiped. It will essentially be an open-top box made of concrete. One side, however, will be made of glass, so that the pool can be observed from below ground. Concrete costs $15 per square meter, and glass costs $100 per square meter. If the volume of the pool must be 1000 cubic meters, what should the dimensions be to minimize the cost of the pool?

4. Double Integrals. Chapter 15

Problem 55. [Midterm 2, Milakis, Winter 2009, 5] Let
\[ D := \{(x, y) \mid 1 \leq x \leq 2, \ln x \leq y \leq e^x\}. \]
Compute the area of \( D \).

Problem 56. [Midterm 2, Perkins, Winter 2009, 3a] Evaluate the following integral:
\[ \int \int_R xy \sin(x^2 y) \, dx \, dy, \quad R := [0, 1] \times [0, \pi/2]; \]

Problem 57. [Midterm 2, Perkins, Winter 2009, 3b] Evaluate the following integral:
\[ \int \int_D y^2 e^{xy} \, dx \, dy, \quad D := \{(x, y) \mid 0 \leq y \leq 3, 0 \leq x \leq y\}. \]

Problem 58. [Final Exam, Spring 2010, 6] Find the volume of the solid that lies under the plane \( 3x + 2y + z = 12 \) and above the rectangle \( R = [0, 1] \times [2, 3] \).

Problem 59. [Final Exam, Spring 2008, 7] Evaluate the integral
\[ \int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} \, dy \, dx. \]

Problem 60. [Final Exam, Spring 2008, 5b] A curve in the \( xy \)-plane, called a cardioid, is determined by the polar equation
\[ r = 1 + \cos \theta. \]
Find the area of the region bounded by the \( x \)-axis and the cardioid \( r = 1 + \cos \theta \) from \( \theta = 0 \) to \( \theta = \pi \).

Problem 61. [Final Exam, Winter 2009, 8] Let
\[ D = \{(x, y) \in \mathbb{R}^2 \mid 4 \leq x^2 + y^2 \leq 4x, \ y \geq 0\} \]
(a) Draw a careful picture for the domain $D$.
(b) Compute the area of $D$.

**Problem 62.** [Final Exam, Winter 2007, 3] (a) Draw a picture of the region $R$ bounded by the circles $x^2 + y^2 = 25$ and $x^2 + y^2 = 16$ in the first quadrant. Label at least two points.
(b) Evaluate the double integral

$$
\iint_R x + \sqrt{x^2 + y^2} \, dx \, dy.
$$

**Problem 63.** [Final Exam, Autumn 2007, 10] Consider the region $R$ bounded by a semi-circle of radius 2, a semi-circle of radius 1, and the $x$-axis (this $R$ lies in the region $\{y > 0\}$). Compute the average value of the function

$$f(x, y) = e^{-x^2 - y^2}$$

over the region $R$.

**Problem 64.** [Final Exam, Spring 2007, 10] (a) Draw the picture of the region $R$ between the curves $r = 2 \cos \theta$ and $r = 2(1 + \cos \theta)$.
(b) Evaluate the area of $R$:

$$A(R) = \iint_R 1 \, dA.$$

**Problem 65.** [Final Exam, Spring 2008, 8] A lamina occupies the region in the $xy$-plane bounded by the lines $x = 1$, $x = 2$, $y = ax$, and $y = 2ax$ for some positive number $a$. The lamina has density function $\rho(x, y) = \frac{1}{x} + \frac{1}{y}$. Find the value of $a$ that minimizes the mass of the lamina.

**Problem 66.** [Midterm 2, Taggart, Spring 2011, 5] The boundary of a lamina consists of the semicircles

$$y = \sqrt{1 - x^2} \quad \text{and} \quad y = \sqrt{25 - x^2}$$

and the portions of the $x$-axis that join them.

The density of the lamina at any point is inversely proportional to its distance from the origin. That is, there is a constant $k$ such that, the density of the lamina at the point $(x, y)$ is

$$\rho(x, y) = \frac{k}{\sqrt{x^2 + y^2}}.$$

Find the center of mass of the lamina.

(You may use the fact that, by symmetry, the center of mass is on the $y$-axis.)

5. Taylor Polynomials and Series

**Problem 67.** [Final Exam, Autumn 2007, 1] Consider the function

$$f(x) := (3 + x)^{1/2}.$$

(a) Find the second Taylor polynomial $T_2$ of $f$ based at $b = 1$. 

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(b) Use the Taylor polynomial you computed above to approximate $\sqrt{3.7}$.
(c) Use Taylor’s inequality to find an upper bound for the error in your approximation above.

**Problem 68.** [Final Exam, Spring 2007, 1] Consider the function $f(x) = x^3 + x$.
(a) Find the second Taylor polynomial $T_2$ of $f$ based at $b = 1$.
(b) Use Taylor’s inequality to find an interval $J$ around $b$ such that the error $|T_2(x) - f(x)|$ is less than 0.001 for all $x$ in $J$.

**Problem 69.** [Final Exam, Winter 2007, 1] Let $f(x) = \frac{1}{5 - x}$, $I = [-2, 2]$, and $b = 0$.
(a) Find the first Taylor polynomial for $f(x)$ based on $b$.
(b) Use Taylor’s inequality to give a bound for the error $|f(x) - T_1(x)|$ on $I$.
(c) Find an integer $n$ such that the error $|f(x) - T_n(x)|$ on $I$ given by Taylor’s inequality is smaller than 0.05 and larger than 0.04.

**Problem 70.** [Final Exam, Winter 2008, 2] Use Taylor’s inequality to find $n$ such that the Taylor polynomial of degree $n$ centered at $x = 0$ for the function $g(x) = e^{2x}$ approximates $g(x)$ with accuracy 0.01 on the interval $[-0.5, 0]$.

**Problem 71.** [Final Exam, Spring 2008, 1] Consider the function $f(x) = \sin(x - 4) + \cos(x - 4) + 4\sqrt{x}$.
(a) Find the second Taylor polynomial $T_2$ of $f(x)$ based at $b = 4$.
(b) Use the second Taylor polynomial $T_2$ to approximate $f(4.1)$.
(c) Use Taylor’s inequality to find an upper bound for the error in your approximation above.

**Problem 72.** [Final Exam, Spring 2010, 3] Let $f(x) = x \ln x$.
(a) Find the second Taylor polynomial $T_2(x)$ for $f(x)$ based at $b = 1$.
(b) Use the Quadratic Approximation Error Bound to find an interval $J$ containing $b$ so that the error bound is at most 0.01.

**Problem 73.** [Final Exam, Winter 2009, 5] Let $f(x) = \ln(e + 3x)$.
(a) Find the Taylor series of the function $f(x)$ centered at $b = 0$.
(b) Find an interval on which the series converges. Justify your answer.

**Problem 74.** [Final Exam, Autumn 2009, 2] Consider the function $f(x) = \ln(1 + 3x^2)$.
(a) Find the Taylor series for the function $f(x) = \ln(1 + 3x^2)$ about $b = 0$. Write your answer in summation notation (hint: no differentiation is necessary).
(b) Find an interval on which the series you just wrote down converges.

**Problem 75.** [Final Exam, Spring 2009, 2] Find the first four terms of the Taylor series based at $b = 0$ for the function
\[ f(x) = \frac{e^{x^2} - 1}{x} + \frac{3}{(1 - x)^2}. \]

**Problem 76.** [Final Exam, Winter 2007, 2] (a) Give the Taylor series for
\[ f(x) = \frac{1}{2x^2 + 1} - \cos(3x) \]
based at $b = 0$. Write your answer using just one sigma sign.
(b) Give the open interval of convergence of the Taylor series in part (a).
(c) Find $T_4(x)$ in expanded notation. Simplify as much as possible.
Solutions

1. Analytic Geometry. Chapter 12

**Problem 1.** (a) \(a\) and \(b\) are orthogonal if and only if the angle \(\theta\) between them is \(\frac{\pi}{2}\), i.e. if and only if \(\cos \theta = 0\). But
\[
\cos \theta = \frac{a \cdot b}{|a||b|}.
\]
Hence \(\cos \theta = 0\) if and only if \(a \cdot b = 0\). In other words, the two vectors \(a, b\) are orthogonal if and only if \(a \cdot b = 0\). But \(a \cdot b = 1 \cdot 0 + 1 \cdot x + x \cdot 1 = 2x\). Hence we have: \(2x = 0, x = 0\). The vectors \(a, b\) are orthogonal if and only if \(x = 0\).

(b) \(\theta = \frac{\pi}{4}\) if and only if \(\cos \theta = \frac{\sqrt{2}}{2}\). We have:
\[
\frac{\sqrt{2}}{2} = \frac{a \cdot b}{|a||b|},
\]
But \(a \cdot b = 2x\) (see above), \(|a| = \sqrt{1^2 + 1^2 + x^2} = \sqrt{2 + x^2}, |b| = \sqrt{0^2 + x^2 + 1^2} = \sqrt{x^2 + 1}\). Hence
\[
\frac{\sqrt{2}}{2} = \frac{2x}{\sqrt{x^2 + 2}\sqrt{x^2 + 1}},
\]
\[
1 = \frac{4x^2}{(x^2 + 2)(x^2 + 1)},
\]
\[
8x^2 = (x^2 + 2)(x^2 + 1), \quad 8x^2 = x^4 + 3x^2 + 2,
\]
\[
x^4 - 5x^2 + 2 = 0, \quad x^2 = \frac{5 \pm \sqrt{17}}{2}
\]
\[
x = \pm \sqrt{\frac{5 \pm \sqrt{17}}{2}}.
\]
(Because \(\frac{5 \pm \sqrt{17}}{2} > 0\).)

**Problem 2.** (a) \(a \cdot b = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = 20, |a| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, |b| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}\). Thus, if \(\theta\) is the angle between \(a\) and \(b\), we have:
\[
\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{20}{\sqrt{14} \sqrt{29}}
\]
(b) \(a \times b = <2 \cdot 4 - 3 \cdot 3, 3 \cdot 2 - 4 \cdot 1, 1 \cdot 3 - 2 \cdot 2> = <-1, 2, -1>\).
(c) This area is \(|PQ \times PR| = |a \times b| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}\).

**Problem 3.** (a) \(b = <5, -7, 2>\). Hence \(a \cdot b = 3 \cdot 5 + (-1) \cdot (-7) + 2 \cdot 2 = 26\).
(b) \(|a| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}, |b| = \sqrt{5^2 + (-7)^2 + 2^2} = \sqrt{78}\). Thus
\[
\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{26}{\sqrt{14} \sqrt{78}}.
\]
\( \theta \approx 38.113^\circ \approx 0.665. \)

(The last number is the radian measure of \( \theta \). You may write any angle either in degrees or radians, but the radians are preferable.)

**Problem 4.** (a) As in Problem 1a, \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \). But
\[
\mathbf{a} \cdot \mathbf{b} = 1 \cdot x + x \cdot 3 + (-4) \cdot 5 = 4x - 20.
\]
Hence we have: \( 4x - 20 = 0, \ x = 5 \).

(b) \( \mathbf{c} := \mathbf{a} \times \mathbf{b} = (-15, -5, 7) \) is orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \). But it is not a unit vector; we have to normalize it: \( |\mathbf{c}| = \sqrt{(-15)^2 + (-5)^2 + 7^2} = \sqrt{299} \), and the vector
\[
\mathbf{n} = \frac{\mathbf{c}}{|\mathbf{c}|} = \left< -\frac{15}{\sqrt{299}}, \ -\frac{5}{\sqrt{299}}, \ \frac{7}{\sqrt{299}} \right>
\]
is a unit vector orthogonal to \( \mathbf{a} \) and \( \mathbf{b} \).

**Problem 5.** Similarly to Problems 1a and 4a, \( \langle x, 3, 2 \rangle \cdot \langle 2, 3, x \rangle = 2 \cdot 2 + 3 \cdot 3 + 2 \cdot x = 4x + 9 \), and these vectors are orthogonal if and only if \( 4x + 9 = 0, \ x = -9/4 \).

**Problem 6.** (a) These planes are not parallel, since their normal vectors, \( \mathbf{n}_1 := \langle 1, 3, 1 \rangle \) and \( \mathbf{n}_2 := \langle 2, 1, -1 \rangle \), are not parallel. One can verify this using their cross product: \( \langle 1, 3, 1 \rangle \times \langle 2, 1, -1 \rangle = \langle 3 \cdot (-1) - 1 \cdot 1, 1 \cdot 1 - 3 \cdot 2, 1 \cdot 1 - 1 \cdot (-1) \rangle = \langle -4, 3, -5 \rangle \neq \mathbf{0} \). Recall that any two nonzero vectors are parallel if and only if their cross product equals \( \mathbf{0} \).

To obtain the parametric equations of the line of intersection, we need: (1) to find a directional vector of this line and (2) to find a certain point on this line.

(1) This line lies on the first plane; therefore, it is orthogonal to the normal vector \( \mathbf{n}_1 \). This line lies on the second plane; therefore, it is orthogonal to the normal vector \( \mathbf{n}_2 \). Their cross product \( \mathbf{n}_1 \times \mathbf{n}_2 \) is also orthogonal to \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \); therefore, we can take this vector as \( \mathbf{v} \), the directional vector of this line. We have already calculated this cross product, it is \( \langle -4, 3, -5 \rangle \).

(2) We have a system of equations:
\[
x + 3y + z = 2, \quad 2x + y - z = -1.
\]
This system contains three variables and two equations. Such systems generally have infinitely many solutions. This one is not an exception. But we need to find only one point (doesn’t matter which one). So set \( z = 0 \), then we have:
\[
x + 3y = 2, \quad 2x + y = -1.
\]
Therefore, \( x = 2 - 3y, \ 2(2 - 3y) + y = -1, \ 4 - 5y = -1, \ 5y = 5, \ y = 1, \ x = -1 \). The point \((-1, 1, 0) \) lies on this line. And we immediately obtain the parametric equations: \( x = -1 - 4t, y = 1 + 3t, z = -5t \).

(b) The angle between these planes is the angle \( \theta \) between the normal vectors, if \( \theta \leq \pi/2 \) (or \( \pi - \theta \), if \( \theta > \pi/2 \). But \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 1 \cdot 2 + 3 \cdot 1 + 1 \cdot (-1) = 4 \), \( |\mathbf{n}_1| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11} \), \( |\mathbf{n}_2| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6} \). Thus
\[
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{4}{\sqrt{11} \sqrt{6}} = \frac{4}{\sqrt{66}}.
\]
Since \( \cos \theta > 0 \), \( \theta < \pi/2 \) and the angle between the planes is \( \theta \). The approximate value of \( \theta \) is \( 61^\circ \).
Problem 7. (a) First of all, these lines are not parallel, since their directional vectors, \( \mathbf{v}_1 := <-1, 1, 1> \) and \( \mathbf{v}_2 := <-2, -1, 3> \), are not parallel. Similarly to Problem 1, this rather obvious fact may be proved by taking the cross product of these vectors:

\[
\mathbf{v}_1 \times \mathbf{v}_2 = <-1 \cdot 3 - 1 \cdot (-1), 1 \cdot (-2) - (-1) \cdot 3, (-1) \cdot (-1) - (-2) \cdot 1> = <-4, 1, 3>.
\]

and observing that this cross product is not equal to the zero vector \( \mathbf{0} \).

Let us rewrite the equations of these lines in parametric form. First line:

\[
x = 2 - t, \quad y = -1 + t, \quad z = t.
\]

Second line:

\[
x = 1 - 2s, \quad y = 3 - s, \quad z = 3s.
\]

To find whether they have any point of intersection, we need to solve the following system of equations:

\[
2 - t = 1 - 2s, \quad -1 + t = 3 - s, \quad t = 3s.
\]

This system contains three equations and two variables. The number of equations exceeds the number of variables, and in general such systems do not have any solution. But this system does have a solution: plugging in \( 3s \) instead of \( t \) in the first two equations, we immediately obtain:

\[
2 - 3s = 1 - 2s, \quad -1 + 3s = 3 - s.
\]

It is easy to check that both of these equations have the same root \( s = 1 \). If they had different roots, the system above would not have any solutions and the lines would not intersect. But these lines do intersect, and the point of intersection is \( \mathbf{r}_2(1) = <-1, 2, 3> \).

(b) Since these lines intersect, there exists a plane that contains these lines. (Two lines are contained in a certain plane if and only if they intersect or are parallel.) This plane contains the point \((-1, 2, 3)\) and is parallel to the vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Hence the following vector is a normal vector to the plane: \( \mathbf{v}_1 \times \mathbf{v}_2 = <-4, 1, 3> \). We can immediately write the equation of this plane:

\[
4(x + 1) + (y - 2) + 3(z - 3) = 0.
\]

Problem 8. First, let us find the line of intersection of these two planes. We need to solve this system of equations:

\[
x + y + z + 5 = 0, \quad 3x + 2y - z + 2 = 0.
\]

Similarly to Problem 1, let us eliminate one of the variables, e.g. \( z \), denoting it as a parameter: \( z = t \). We obtain:

\[
x + y = -5 - t, \quad 3x + 2y = -2 + t.
\]

Hence

\[
x = 3x + 2y - 2(x + y) = -2 + t - 2(-5 - t) = 8 + 3t,
\]

\[
y = -5 - t - y = -5 - t - (8 + 3t) = -13 - 4t.
\]

Thus, parametric equations of the line of intersection:

\[
x = 8 + 3t, \quad y = -13 - 4t, \quad z = t.
\]
Hence the distance from the point \((1 + t, 2, 1)\) to the line is 3. Thus, there are two points on the line at distance 3 from this plane: \(\langle 1, 2, 1 \rangle\) and \(\langle 1, 2, -1 \rangle\). This plane contains the point \(P(1, 2, 1)\) and its normal vector is \(\langle 19, 10, -17 \rangle\). Thus, its equation is
\[
19(x - 1) + 10(y - 2) - 17(z - 1) = 0.
\]

**Problem 9.** (a) A line and a plane are parallel if and only if the directional vector of the line is orthogonal to the normal vector of the plane. But \(\langle 1, 3, -1 \rangle\) is the directional vector of this line, and \(\langle 1, 2, 2 \rangle\) is the normal vector of this plane. Their dot product is \(\langle 1, 3, -1 \rangle \cdot \langle 1, 2, 2 \rangle = 1 \cdot 1 + 3 \cdot 2 + (-1) \cdot 2 = 5 \neq 0\), hence they are not orthogonal; and the given line and plane are not parallel.

(b) The distance from the point \((x_0, y_0, z_0)\) to the plane \(x + 2y + 2z + 2 = 0\) is equal to
\[
d = \frac{|x_0 + 2y_0 + 2z_0 + 2|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{|x_0 + 2y_0 + 2z_0 + 2|}{3}.
\]

The parametric equations of the line:
\[
x = 1 + t, \ y = 1 + 3t, \ z = 1 - t.
\]
Hence the distance from the point \((1 + t, 1 + 3t, 1 - t)\) to the plane is
\[
\frac{1}{3}|(1 + t) + 2(1 + 3t) + 2(1 - t) + 2| = \frac{1}{3}|7 + 5t|.
\]
It is equal to 3 if and only if
\[
|7 + 5t| = 9, \ 7 + 5t = \pm 9, \ 5t = 2 \text{ or } -16, \ t = \frac{2}{5} \text{ or } -\frac{16}{5}.
\]
Thus, there are two points on the line at distance 3 from this plane:
\[
\left(\frac{7}{5}, \frac{11}{5}, \frac{3}{5}\right), \left(\frac{-11}{5}, \frac{-43}{5}, \frac{21}{5}\right).
\]

**Problem 10.**
1. FALSE. For ANY vector \(v\), we have: \(v \times v = 0\).
2. TRUE. If \(v \cdot v = 0\), then \(|v|^2 = 0\), \(|v| = 0\), and the only vector with magnitude 0 is the zero vector.
3. FALSE. \(v \cdot w\) is a scalar, and you cannot cross-multiply a vector by a scalar.
4. TRUE.
5. FALSE. \(v \cdot w\) is a scalar.
6. TRUE.
7. TRUE. \((v - w) \cdot (v + w) = v \cdot v - v \cdot w + v \cdot w - w \cdot w = v \cdot v - w \cdot w\), because \(v \cdot w = w \cdot v\).
8. FALSE. \((v - w) \times (v + w) = v \times v - w \times v + v \times w - w \times w = v \times v - w \times w + 2v \times w \neq v \times v - w \times w\), because \(v \times w\) is not necessarily \(0\).
Problem 11. Solution 1. Suppose \( P = (x,y,z) \). Then \( \mathbf{AP} = <x-1,y-2,z-3> \), \( \mathbf{OP} = <x,y,z> \), \( \mathbf{OA} = <1,2,3> \). Therefore, \( \mathbf{AP} \cdot \mathbf{OP} = x(x-1)+y(y-2)+z(z-3) \), \( \mathbf{OA} \cdot \mathbf{OP} = x+2y+3z \), \( |\mathbf{OA}|^2 = 1^2+2^2+3^2 \). We get:

\[
\begin{align*}
x(x-1) + y(y-2) + z(z-3) - (x+2y+3z) &= 2 - 1^2 - 2^2 - 3^2, \\
(x^2 - 2x + 1^2) + (y^2 - 4y + 2^2) + (z^2 - 6z + 3^2) &= 2, \\
(x-1)^2 + (y-2)^2 + (z-3)^2 &= 2.
\end{align*}
\]

Thus, this is a sphere with center \((1,2,3)\) and radius \( \sqrt{2} \).

Solution 2. Let \( \mathbf{a} := \mathbf{OA} \), \( \mathbf{r} := \mathbf{AP} \). Then \( \mathbf{OP} = \mathbf{a} + \mathbf{r} \), and we can rewrite this equation as:

\[
\begin{align*}
\mathbf{r} \cdot (\mathbf{a} + \mathbf{r}) - \mathbf{a} \cdot (\mathbf{a} + \mathbf{r}) &= 2 - |\mathbf{a}|^2, \\
\mathbf{r} \cdot \mathbf{a} + \mathbf{r} \cdot \mathbf{r} - \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{r} &= 2 - \mathbf{a} \cdot \mathbf{a}, \\
\mathbf{r} \cdot \mathbf{r} &= 2.
\end{align*}
\]

Thus we immediately get that this is a sphere with center \( A \) and radius \( \sqrt{2} \).

Problem 12. (a) The distance from \( p = (x,y,z) \) to the plane \( y = 5 \) is equal to \( |y-5| \). The distance from \( p \) to the line \( y = 1, z = 2 \) is equal to \( \sqrt{(y-1)^2 + (z-2)^2} \). They are equal if and only if

\[
|y-5| = \sqrt{(y-1)^2 + (z-2)^2}, \quad (y-5)^2 = (y-1)^2 + (z-2)^2,
\]

\[
y^2 - 10y + 25 = y^2 - 2y + 1 + (z-2)^2 \]

\[
(z-2)^2 + 8y - 24 = 0.
\]

Suppose we did not have the \( x \)-coordinate. Then this would be a parabola which "faces" along the negative \( y \)-axis. In fact, this is a "parabolic cylinder" parallel to the \( x \)-axis. (Any surface which does not have \( x \) in its equation is parallel to the \( x \)-axis.)

(b) Let us plug \( z = 6 \) in the equation of \( S: (6-2)^2 + 8y - 24 = 0, 8y - 8 = 0, y = 1 \). The equations of this trace are: \( z = 6, y = 1 \). This is a line parallel to the \( x \)-axis.

Problem 13. It is fairly obvious that the diagonals intersect at the center of the cube: \( M = (1/2, 1/2, 1/2) \). Of course, one can verify this by the following direct computation (but you do not need to write this at an exam).

The equations of the line containing the first diagonal are \( x = t, y = t, z = t \) (since it passes through the point \( (0, 0, 0) \) and its directional vector is \( <1-0,1-0,1-0> = <1,1,1> \). Similarly, the equations of the line containing the second diagonal are \( x = 1-s, y = s, z = s \). They intersect at the point where \( t = s \) and simultaneously \( t = 1-s \), i.e. \( s = 1-s, s = 1/2 \); this point is \((1-1/2 = 1/2, 1/2, 1/2) \).

Let \( O = (0,0,0) \), \( A = (1,0,0) \). The angle between these diagonals is the angle \( \theta = \angle OMA \), if \( \theta \leq \pi/2 \), or \( \pi - \theta \), if \( \theta > \pi/2 \). Therefore, we need to find \( \theta \). But

\[
\cos \theta = \frac{\mathbf{MO} \cdot \mathbf{MA}}{|\mathbf{MO}||\mathbf{MA}|}.
\]

Since \( \mathbf{MA} = <1/2,-1/2,-1/2> \), \( \mathbf{MO} = <-1/2,-1/2,-1/2> \), we have: \( \mathbf{MO} \cdot \mathbf{MA} = 1/4 \), \( |\mathbf{MO}| = |\mathbf{MA}| = \sqrt{3}/2 \). Thus

\[
\cos \theta = \frac{1/4}{(\sqrt{3}/2) \cdot (\sqrt{3}/2)} = \frac{1/4}{3/4} = \frac{1}{3}.
\]

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Since \(\cos \theta > 0, \theta < \pi/2\) and the angle between the two diagonals is \(\cos^{-1}(1/3) \approx 1.23 \approx 70.5^\circ\).

**Problem 14.** Let \(A = (2,0,0), B = (4,3,5), C = (0,1,3)\). Let \(a = AB =< 2,3,5 >, b = BC =< -4,-2,-2 >, c = CA =< 2,-1,-3 >\). Then the triangle \(ABC\) is right if and only if the angle between some of the vectors \(a, b, c\) is \(\pi/2\), i.e. if and only if \(a \cdot b = 0\) or \(a \cdot c = 0\) or \(b \cdot c = 0\). If one computes these dot products, he finds that \(b \cdot c = 0\), i.e. this is indeed a right triangle (with \(\angle C = \pi/2\)).

**Problem 15.** (a) It suffices to find a normal vector to the plane; then we can immediately write the equation, because we already have a point \(P = (0,2,5)\) on this plane. A normal vector can be obtained as a cross product of two vectors on this plane. The first vector is \(v_1 =< 3,2,1 >\) (the directional vector of this line; since the line lies on this plane, this directional vector also lies on this plane). The second vector is \(v_2 = PQ\), where \(Q\) is any point on this line. Take, e.g. \(t = 0\) and get \(Q = (1,2,3)\). Then \(v_2 = PQ = < 1,0,-2 >\).

And the normal vector is \(n = v_1 \times v_2 = < -4,7,-2 >\). So the equation of this plane is

\[-4x + 7(y - 2) - 2(z - 5) = 0.\]

(b) The line of intersection lies on the first plane; therefore, it is orthogonal to the normal vector \(n_1 =< 2,0,-1 >\) of this plane. The line of intersection lies on the second plane; therefore, it is orthogonal to the normal vector \(n_2 =< 1,1,1 >\) of this plane. The vector \(v = n_1 \times n_2 =< 1, -3, 2 >\) is also orthogonal to both normal vectors; therefore, it is a directional vector of this line.

It suffices to find some point on this line. Set \(z = 0\) and solve the system of equations
\[2x = 0, \ x + y = 1.\]
We get: \(x = 0, y = 1\). So \((0,1,0)\) lies on both planes, therefore, on this line. Thus, the equation of this line is
\[x = t, \ y = 1 - 3t, \ z = 2t.\]

**Problem 16.** (a) The vectors \(a = PQ =< -1,2,-1 >\) and \(b = PR =< 1,-1,1 >\) lie on this plane. Therefore, the following vector is normal: \(n = a \times b = < 1,0,-1 >\).

(b) The area of the triangle \(PQR\) is half of the area of the parallelogram based on the vectors \(a, b\). The area of this parallelogram is \(|a \times b| = \sqrt{2}\). Thus, the area of this triangle is \(\sqrt{2}/2\).

(c) This plane passes through the point \((2,1,5)\) and has a normal vector \(< 1,0,-1 >\). Therefore, its equation is
\[1 \cdot (x - 2) + 0 \cdot (y - 1) + (-1) \cdot (z - 5) = 0, \ x - z + 3 = 0.\]
And this point \(T(0,3,3)\) satisfies this equation; therefore, it lies on the plane which passes through \(P, Q\) and \(R\).

**Problem 17.** (a) \(a = AB =< -3,4,0 >, \ b = AC =< -3,0,1 >\). The area of this triangle is
\[
\frac{1}{2} |a \times b| = \frac{1}{2} |< 4,3,12 >| = \frac{1}{2} \sqrt{4^2 + 3^2 + 12^2} = \frac{13}{2},
\]
because \(n = a \times b = < 4,3,12 >\).

(b) Let us find the equations of the lines \(CH\) and \(AB\). The directional vector of the line \(AB\) is \(AB =< -3,4,0 >\). This line passes through the point \(A = (3,0,0)\). Therefore, its parametric equation is
\[x = 3 - 3t, \ y = 4t, \ z = 0.\]
The line \( CH \) lies on the plane that passes through the points \( A, B, C \) and therefore is orthogonal to the normal vector \( \mathbf{n} = \langle 4, 3, 12 \rangle \). This line is also orthogonal to the vector \( \mathbf{AB} = \langle -3, 4, 0 \rangle \). Therefore, its directional vector is

\[
\mathbf{n} \times \mathbf{AB} = \langle -48, -36, 25 \rangle .
\]

This line passes through the point \( C = (0, 0, 1) \); therefore, its equation is

\[
x = -48s, \ y = -36s, \ z = 25s + 1 .
\]

The point \( H \) is the point of intersection of these lines; to find it, we need to solve the system

\[
-48s = 3 - 3t, \quad -36s = 4t, \quad 25s + 1 = 0 .
\]

\( s = -1/25 \) from the third equation, so \( x = 48/25, \ y = 36/25, \ z = 0 \) are the coordinates of \( H \).

**Problem 18.** (a) This plane contains vectors

\[
(1, 0, 0) - (0, 1, 0) = \langle 1, -1, 0 \rangle
\]

and

\[
(1, 0, 0) - (0, 0, 1) = \langle 1, 0, -1 \rangle .
\]

Hence their cross product is a normal vector:

\[
\mathbf{n} = \langle 1, -1, 0 \rangle \times \langle 1, 0, -1 \rangle = \langle 1, 1, 1 \rangle .
\]

This plane contains the point \( (1, 0, 0) \), hence its equation is

\[
(x - 1) \cdot 1 + (y - 0) \cdot 1 + (z - 0) \cdot 1 = 0, \quad x + y + z - 1 = 0 .
\]

(b) The distance from the point \( (2, 0, 0) \) to this plane is

\[
\frac{|2 + 0 + 0 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}} .
\]

**Problem 19.** Suppose the vector \( \langle x, y, z \rangle \) is orthogonal to \( \mathbf{a} = \langle 11, 3, -5 \rangle \). Then their dot product is 0, so

\[
11x + 3y - 5z = 0 .
\]

There are many solutions to this equation; e.g. let \( x = 0, y = 5, z = 3 \). So the vector \( \mathbf{v} = \langle 0, 5, 3 \rangle \) is orthogonal to \( \mathbf{a} \). But it does not have length 7; it has length \( \sqrt{5^2 + 3^2} = \sqrt{34} \). So the vector \( (1/\sqrt{34})\mathbf{v} \) has length 1, and the vector

\[
\frac{7}{\sqrt{34}} \mathbf{v} = \langle 0, \frac{35}{\sqrt{34}}, \frac{21}{\sqrt{34}} \rangle
\]

has length 7 and the same direction as \( \mathbf{v} \); therefore, it is orthogonal to \( \mathbf{a} \).

We might as well take other solutions to the equation above, e.g. \( x = -3, y = 11, z = 0 \).

**Problem 20.** Let \( P = (1, 2, 3) \), \( Q = (-2, 5, 7) \), and \( R = (-5, 8, 11) \). These points lie on the same line if and only if the vectors \( \mathbf{a} = \overrightarrow{PQ} = \langle -3, 3, 4 \rangle \) and \( \mathbf{b} = \overrightarrow{QR} = \langle -3, 3, 4 \rangle \) are parallel.
But, obviously, they are equal, so they are parallel. (You might as well take the cross product in case of different numbers, when it is not so obvious.)

**Problem 21.** Let \( \mathbf{v} = <3, 4, -1> \) and \( \mathbf{w} = <5, 2, 8> \). Then \( \mathbf{v} \cdot \mathbf{w} = 3 \cdot 5 + 4 \cdot 2 + (-1) \cdot 8 = 15 \), and the magnitudes of these vectors are: \( |\mathbf{v}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26} \), \( |\mathbf{w}| = \sqrt{5^2 + 2^2 + 8^2} = \sqrt{93} \). So if \( \theta \) is the angle between these vectors, we have:

\[
\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{15}{\sqrt{26}\sqrt{93}}, \quad \theta = \arccos \left( \frac{15}{\sqrt{26}\sqrt{93}} \right).
\]

**Problem 22.** Let \( A = (x, y, z) \) be a point on this surface. The distance from \( A \) to the \( x \)-axis is \( \sqrt{y^2 + z^2} \). The distance from \( A \) to the \( z \)-axis is \( \sqrt{x^2 + y^2} \). So the equation of the surface is

\[
\sqrt{x^2 + y^2} = 2\sqrt{y^2 + z^2}.
\]

Simplify this equation:

\[
x^2 + y^2 = 4(y^2 + z^2), \quad x^2 = 3y^2 + 4z^2.
\]

Take \( y = 0, z = 1 \); then \( x^2 = 4 \), and we can take \( x = 2 \); so \((2, 0, 1)\) is a point on this surface, other than the origin. This surface is a cone.
2. Differential Geometry. Chapters 10 and 13

Problem 23. \(3x^2 + y^2 = 6 - 3x^2 - y^2, 3x^2 + y^2 = 3, z = 3x^2 + y^2 = 3\). So we already know \(z\).
How to parametrize \(3x^2 + y^2 = 3\)?
\[x^2 + \left( \frac{y}{\sqrt{3}} \right)^2 = 1.\]
So there exists a \(t\) such that \(x = \cos t, y/\sqrt{3} = \sin t\). We use the fact that for every pair of real numbers \((p, q)\) such that \(p^2 + q^2 = 1\) there exists a real number \(t\) such that \(p = \cos t, q = \sin t\).
The proof of this fact is easy: just take the point \(A = (p, q)\) on the unit circle and let \(t\) be the angle between the positive \(x\)-axis and the vector \(\langle p, q \rangle = OA\), where \(O\) is the origin.
Therefore,
\[x = \cos t, \quad y = \sqrt{3}\sin t, \quad z = 3\]
is a desired parametrization.

Problem 24. We use the same method as in Problem 1. For this problem, rewrite the first equation as
\[\left( \frac{2x}{3} \right)^2 + \left( \frac{z-1}{3} \right)^2 = 1.\]
Take \(p = 2x/3, q = (z-1)/3\). There exists \(t\) such that \(2x/3 = \cos t, (z-1)/3 = \sin t\), or, after simple transformations,
\[x = \frac{3}{2} \cos t, \quad z = 1 + 3 \sin t.\]
Finally,
\[y = 3x^2 = 3 \left( \frac{3}{2} \right)^2 \cos^2 t = \frac{27}{4} \cos^2 t.\]
Thus,
\[\mathbf{r}(t) = < \frac{3}{2} \cos t, \frac{27}{4} \cos^2 t, 1 + 3 \sin t >.\]

Problem 25. First, let us find the point of intersection of these curves. We must solve the system of equations
\[t^3 = s - 4, \quad 2t^2 + 1 = s - 3, \quad 2t + 3 = s - 1.\]
Subtract the third equation from the second and obtain \(2t^2 - 2t - 2 = -2, 2t^2 - 2t = 0, 2t(t-1) = 0,\) so \(t = 0\) or \(t = 1\). Plug in \(t = 0\): \(0 = s - 4, 1 = s - 3, 3 = s - 1,\) so \(s = 4\). Plug in \(t = 1\): \(1 = s - 4, 3 = s - 3, 5 = s - 1,\) so \(s = 5\) and \(s = 6\) - this is impossible, so the case \(t = 1\) does not give us any solution. Thus, the only point of intersection is at \(t = 0, s = 4\). (This is \(x = 0, y = 1, z = 3.\))

Since \(\mathbf{r}_1'(t) = < 3t^2, 4t, 2 >\), the tangent vector to the first curve at this point of intersection is \(\mathbf{a} = \mathbf{r}_1'(0) = < 0, 0, 2 >.\) Since \(\mathbf{r}_2'(s) = < 1, 1, 1 >\), the tangent vector to the second curve at this point of intersection is \(\mathbf{b} = \mathbf{r}_2'(4) = < 1, 1, 1 >.\) The angle between the curves is the angle between \(\mathbf{a}\) and \(\mathbf{b}\). If \(\theta\) is this angle, then
\[\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| \cdot ||\mathbf{b}||}.\]
But \(\mathbf{a} \cdot \mathbf{b} = 2, ||\mathbf{a}|| = 2, ||\mathbf{b}|| = \sqrt{3},\) so \(\cos \theta = 1/\sqrt{3},\) and \(\theta = \arccos(1/\sqrt{3}).\)
Problem 26. The lowest point is the point with the minimal \( y \). Let us find the minimum point of \( y(t) = \sin t - \cos t \). We can do this without calculus. Indeed,

\[
y(t) = \sqrt{2} \left( \frac{1}{\sqrt{2}} \sin t - \frac{1}{\sqrt{2}} \cos t \right) = \sqrt{2} \left( \cos \frac{\pi}{4} \sin t - \sin \frac{\pi}{4} \cos t \right) = \sqrt{2} \sin \left( t - \frac{\pi}{4} \right).
\]

This function attains its (global) minimum when \( t - \pi/4 = -\pi/2, \ t = -\pi/4 \). Thus, the lowest point is

\[
x(-\pi/4) = 2 \cos(-\pi/4) + \sin(-\pi/4) = 2 \cos(\pi/4) - \sin(\pi/4) = \sqrt{2}/2, \\
y(-\pi/4) = \sin(-\pi/4) - \cos(-\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2}.
\]

Problem 27. Let us find the length of this curve.

\[x'(t) = e^t(\cos t - \sin t), \quad y'(t) = e^t(\sin t + \cos t)\]

Therefore,

\[x'(t)^2 + y'(t)^2 = e^{2t} ((\cos t - \sin t)^2 + (\sin t + \cos t)^2) = e^{2t} (\cos^2 t - 2 \cos t \sin t + \sin^2 t + \sin^2 t + 2 \cos t \sin t - 2 \cos^2 t)\]

And \( \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{2}e^t \), so the length of this curve is

\[
\int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^{2\pi} \sqrt{2}e^t dt = \sqrt{2}(e^{2\pi} - 1).
\]

Problem 28. The curve is given by the equation

\[r(t) = <t, \ln \cos t, 0>.
\]

Therefore,

\[r'(t) = <1, -\frac{\sin t}{\cos t}, 0> = <1, -\tan t, 0>,
\]

\[r''(t) = <0, -\frac{1}{\cos^2 t}, 0>.
\]

Then we get

\[r'(t) \times r''(t) = <0, 0, -\frac{1}{\cos^2 t}>, \quad |r'(t) \times r''(t)| = \frac{1}{\cos^2 t}.
\]

Also, \( |r'(t)| = \sqrt{1 + \tan^2 t} = \sqrt{1/\cos^2 t} = 1/|\cos t| \), and the curvature is

\[k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{1/\cos^2 t}{1/|\cos t|^3} = 1/|\cos t|^2 = |\cos t|.
\]

The maximum value of this function is 1.

Problem 29. (b) The parametric equations of this curve in Cartesian coordinates are

\[x = r \cos \theta = 2 \cos \theta(1 - \cos \theta) = 2 \cos \theta - 2 \cos^2 \theta, \]

\[y = r \sin \theta = 2 \sin \theta(1 - \cos \theta) = 2 \sin \theta - 2 \sin \theta \cos \theta.
\]
The directional vector of the tangent line at point \( \theta \) is

\[
< x'(\theta), y'(\theta) >= < -2 \sin \theta + 4 \cos \theta \sin \theta, 2 \cos \theta - 2 \cos^2 \theta + 2 \sin^2 \theta >.
\]

This line is horizontal if and only if the y-component of this directional vector is 0, i.e.

\[
2 \cos \theta - 2 \cos^2 \theta + 2 \sin^2 \theta = 0.
\]

Let us solve this equation. Denote \( u := \cos \theta \) and observe that \( \sin^2 \theta = 1 - u^2 \). So

\[
u - u^2 + (1 - u^2) = 0, 2u^2 - u - 1 = 0, 2(u-1)(u+1/2) = 0,
\]

and either \( u = \cos \theta = 1 \) or \( u = \cos \theta = -1/2 \). In the first case, \( \theta = 0 \) and \( x = 0, y = 0 \). But in fact, this case is invalid, because \( x'(0) = y'(0) = 0 \), and this means that the tangent line simply does not exist! In the second case, \( \theta = \pm 2\pi/3 \) and \( x = 2u - 2u^2 = -3/2, y = \pm(\sqrt{3} + \sqrt{3}/2) = \pm 3\sqrt{3}/2 \). So there are two such points: \( (-3/2, 3\sqrt{3}/2), (-3/2, -3\sqrt{3}/2) \).

**Problem 30.** (a) \( r'(t) = < 2t, -3t^2 \sin t^3, 3t^2 \cos t^3 > \). So

\[
|r'(t)| = \sqrt{4t^2 + (3t^2)^2 \cos^2 t^3 + (3t^2)^2 \sin^2 t^3} = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}.
\]

The length from 0 to \( t \) is

\[
s(t) = \int_0^t |r'(u)|du = \int_0^t u\sqrt{4 + 9u^2}du = \int_4^{4 + 9t^2} \frac{dv}{18\sqrt{v}}
\]

(we changed variables \( v = 4 + 9u^2, dv = 18udu \)). Therefore,

\[
s(t) = \frac{1}{18} \left( \frac{v^{3/2}}{3/2} \right)_4^{4 + 9t^2} = \frac{1}{27} \left( (4 + 9t^2)^{3/2} - 4^{3/2} \right) = \frac{1}{27} \left( (4 + 9t^2)^{3/2} - 8 \right) .
\]

In particular,

\[
s(2\pi) = \frac{1}{27} \left( (4 + 36\pi^2)^{3/2} - 8 \right) .
\]

(b) We have: \( 27s + 8 = (4 + 9t^2)^{3/2}, (27s + 8)^{2/3} - 4 = 9t^2 \), so

\[
t = \frac{1}{3}((27s + 8)^{2/3} - 4)^{1/2}.
\]

Thus,

\[
r(t) = < \frac{1}{9}((27s + 8)^{2/3} - 4), \cos \frac{1}{27}((27s + 8)^{2/3} - 4)^{3/2}, \sin \frac{1}{27}((27s + 8)^{2/3} - 4)^{3/2} >.
\]

**Problem 31.** (a) \( x = r \cos \theta = 4 \cos^2 \theta + \cos \theta \sin \theta, y = r \sin \theta = 4 \cos \theta \sin \theta + \sin^2 \theta \). So

\[
x = 2(\cos 2\theta + 1) + \frac{1}{2} \sin 2\theta, \quad y = 2 \sin 2\theta + \frac{1 - \cos 2\theta}{2}.
\]

And we have:

\[
x - 2 = 2 \cos 2\theta + \frac{1}{2} \sin 2\theta, \quad y - \frac{1}{2} = 2 \sin 2\theta - \frac{1}{2} \cos 2\theta,
\]
\[(x - 2)^2 + (y - \frac{1}{2})^2 = (2 \cos 2\theta + \frac{1}{2} \sin 2\theta)^2 + (2 \sin 2\theta - \frac{1}{2} \cos 2\theta)^2 = \frac{17}{4}\]

(just expand it out and use the trig identity \(\sin^2 + \cos^2 = 1\)).

(b) The coordinates of the point on the curve where \(\theta = \pi/4\) are \(x = 5/2, y = 5/2\) (after plugging in \(\theta = \pi/4\)). Since

\[x'(\theta) = -8 \cos \theta \sin \theta + \cos^2 \theta - \sin^2 \theta, \quad y'(\theta) = 4(\cos^2 \theta - \sin^2 \theta) + 2 \sin \theta \cos \theta,\]

the tangent vector at this point is

\[< x'(\pi/4), y'(\pi/4) > = < -4, 1 > .\]

This vector is a directional vector for the tangent line; and this line passes through the point \((5/2, 5/2)\). Therefore,

\[x = -4t + 5/2, \quad y = t + 5/2\]

is the parametric equation of this tangent line.

**Problem 32.**

\[\mathbf{r}'(t) = \left< -\frac{4t}{(1 + t^2)^2}, \frac{2(t^2 + 1) - 2t \cdot 2t}{(t^2 + 1)^2}, 0 \right>, \quad 0 < -\frac{4t}{(t^2 + 1)^2}, \frac{2(1 - t^2)}{(t^2 + 1)^2}, 0 > .\]

\[|\mathbf{r}'(t)| = \sqrt{\left( -\frac{4t}{(t^2 + 1)^2} \right)^2 + \left( \frac{2(t^2 - 1)}{(t^2 + 1)^2} \right)^2 + 0^2} = \sqrt{\frac{16t^2 + 4(t^2 - 1)^2}{(1 + t^2)^4}} = \sqrt{\frac{4t^4 + 8t^2 + 4}{(1 + t^2)^4}} = \sqrt{\frac{4(t^2 + 1)^2}{(1 + t^2)^4}} = \frac{2}{t^2 + 1}.\]

The point \((1, 0, 1)\) corresponds to \(t = 0\). Indeed, compare the second components: \(2t/(t^2 + 1) = 0\) if and only if \(t = 0\). The arclength from \(t = 0\) to \(t\) is

\[s(t) = \int_0^t |\mathbf{r}'(u)|du = \int_0^t \frac{2}{u^2 + 1}du = 2 \arctan t.\]

So \(t = \tan(s(t)/2)\). And

\[\mathbf{r}(t) = \left< \frac{2}{\tan^2(s/2)} - 1, \frac{2\tan(s/2)}{\tan^2(s/2) + 1}, 1 \right> = \left< 2 \cos^2 \frac{s}{2} - 1, 2 \sin \frac{s}{2} \cos \frac{s}{2}, 1 \right> = \left< \cos s, \sin s, 1 \right> .\]

**Problem 33.** A normal plane at any point of a curve is the plane which passes through this point and has a tangent vector to this curve as its normal vector. But \(\mathbf{r}'(t) = \left< 1, 2t, 3t^2 \right>\). Hence:

(a) at the point where \(t = 1\) (this point is \((1, 1, 1)\)) we have \(\mathbf{r}'(1) = \left< 1, 2, 3 \right>\), and the equation of the plane is

\[1 \cdot (x - 1) + 2 \cdot (y - 1) + 3 \cdot (z - 1) = 0, \quad x + 2y + 3z = 6.\]

(b) at the point \((-1, 1, -1)\) (where \(t = -1\)) we have \(\mathbf{r}'(1) = \left< 1, -2, 3 \right>\), and the equation of the plane is

\[1 \cdot (x + 1) + (-2) \cdot (y - 1) + 3 \cdot (z + 1) = 0, \quad x - 2y + 3z + 6 = 0.\]
(c) The normal vectors to these planes are \( \mathbf{n}_1 = <1, 2, 3> \) and \( \mathbf{n}_2 = <1, -2, 3> \), so the directional vector of the intersection line is given by \( \mathbf{n}_1 \times \mathbf{n}_2 = <12, 0, -4> \).

Indeed, this line lies on the first plane; therefore, it is orthogonal to the first normal vector \( \mathbf{n}_1 \). Similarly, this line lies on the second plane; therefore, it is orthogonal to \( \mathbf{n}_2 \). Recall that the cross product \( \mathbf{n}_1 \times \mathbf{n}_2 \) is also orthogonal to \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \); and so we can take this cross product as a directional vector of this line.

But \( <12, 0, -4> = <4, 3, 0, -1> \); so we can also take \( <3, 0, -1> \) as a directional vector. Let us find any point which lies on this line, or, in other words, satisfies the equations

\[
x + 2y + 3z = 6, \quad x - 2y + 3z + 6 = 0.
\]

Set up \( z = 0 \). Then we have:

\[
x + 2y = 6, \quad x - 2y = -6.
\]

Sum these equations: \( 2x = 0, \ x = 0, \ 2y = 6 - x = 0, \ y = 3. \) So this point is \( (0, 3, 0) \). And the parametric equation of the line of intersection are

\[
x = 3t, \ y = 3, \ z = -t.
\]

**Problem 34.** (a) Use the fundamental theorem of calculus for the vector function \( \mathbf{r}'(t) \):

\[
\mathbf{r}'(t) = \int_0^t \mathbf{r}''(s)ds + \mathbf{r}'(0) = \int_0^t (2i + \cos s \mathbf{j} + \sin s \mathbf{k})ds - \mathbf{k} = \mathbf{i} \int_0^t 2ds + \mathbf{j} \int_0^t \cos sds + \mathbf{k} \int_0^t \sin sds - \mathbf{k} = 2\mathbf{i} + \sin t\mathbf{j} + (1 - \cos t)\mathbf{k} - \mathbf{k} = 2\mathbf{i} + \sin t\mathbf{j} - \cos t\mathbf{k}.
\]

Similarly,

\[
\mathbf{r}(t) = \int_0^t \mathbf{r}'(s)ds + \mathbf{r}(0) = \int_0^t (2s\mathbf{i} + \sin s\mathbf{j} - \cos s\mathbf{k})ds + \mathbf{i} + \mathbf{j} = \mathbf{i} \int_0^t 2ds + \mathbf{j} \int_0^t \sin sds - \mathbf{k} \int_0^t \cos sds + \mathbf{i} + \mathbf{j} = (t^2 + 1)\mathbf{i} + (2 - \cos t)\mathbf{j} - \sin t\mathbf{k}.
\]

(b) The curvature:

\[
\mathbf{r}'(1) = 2\mathbf{i} + \sin 1\mathbf{j} - \cos 1\mathbf{k} = <2, \sin 1, -\cos 1>, \mathbf{r}''(1) = <2, \cos 1, \sin 1>,
\]

\[
\mathbf{r}'(1) \times \mathbf{r}''(1) = \mathbf{i} - 2(\cos 1 + \sin 1)\mathbf{j} + 2(\cos 1 - \sin 1)\mathbf{k},
\]

\[
|r'(1) \times r''(1)| = \sqrt{1 + 4(\cos^2 1 + \sin^2 1)} = \sqrt{5},
\]

and the curvature at the point \( t = 1 \) is equal to

\[
\frac{|r'(1) \times r''(1)|}{|r'(1)|^3} = \frac{3}{\sqrt{5}^3} = \frac{3\sqrt{5}}{25}.
\]
Problem 35. The particle passes through $yz$-plane when $x = 0$, i.e. $3t - 6 = 0$, $t = 2$. The velocity is
\[ \mathbf{r}'(t) = <3, 6t^2 - 5, -2t >. \]
So $\mathbf{r}'(2) = <3, 24 - 5, -4 > = <3, 19, -4 >$, and the speed (= the absolute value of velocity) at this moment is equal to $\sqrt{3^2 + 19^2 + (-4)^2} = \sqrt{386}$.

Problem 36. (a) We shall use formulas from the end of Section 13.3, Stewart. First of all, $\mathbf{r}'(t) = <2t, 2, 1 >$. Hence $|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 2^2 + 1^2} = \sqrt{4t^2 + 5},$ 
\[ \mathbf{T}(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}'(t)|} = <\frac{2t}{\sqrt{4t^2 + 5}}, \frac{2}{\sqrt{4t^2 + 5}}, \frac{1}{\sqrt{4t^2 + 5}} > \]
This vector is parallel to $\mathbf{r}$. Hence it is parallel to the plane $x + y + z = 0$ if and only if the vector $\mathbf{r}$ is parallel to this plane. And this condition is equivalent to the following: $\mathbf{r}$ is orthogonal to the vector $<1, 1, 1 >$, which is normal to this plane. These vectors are orthogonal if and only if their dot product equals 0, i.e.
\[ 2t \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 0, \quad 2t + 3 = 0, \quad t = -3/2. \]

Let us compute the normal vector:
\[ \frac{d}{dt} \left( \frac{2t}{\sqrt{4t^2 + 5}} \right) = \frac{(2t)' \sqrt{4t^2 + 5} - 2t(t' \sqrt{4t^2 + 5})'}{4t^2 + 5} = \frac{2\sqrt{4t^2 + 5} - 2t \frac{8t}{2\sqrt{4t^2 + 5}}}{4t^2 + 5} = \]
\[ = \frac{(4t^2 + 5) - 8t^2}{(4t^2 + 5)^{3/2}} = \frac{10}{(4t^2 + 5)^{3/2}}. \]
\[ \frac{d}{dt} \left( \frac{1}{\sqrt{4t^2 + 5}} \right) = \left( -\frac{1}{2} \right) \frac{(4t^2 + 5)'}{(4t^2 + 5)^{3/2}} = -\frac{4t}{(4t^2 + 5)^{3/2}}. \]
Hence, differentiating each component, we obtain:
\[ \mathbf{T}'(t) = <\frac{10}{(4t^2 + 5)^{3/2}}, -\frac{8t}{(4t^2 + 5)^{3/2}}, -\frac{4t}{(4t^2 + 5)^{3/2}} >. \]
At the point $t = -3/2$, we have $4t^2 + 5 = 4(-3/2)^2 + 5 = 14$, and
\[ \mathbf{T}(-3/2) = <\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} > = \frac{1}{\sqrt{14}} < -3, 2, 1 >, \]
\[ \mathbf{T}'(-3/2) = <\frac{10}{14^{3/2}}, \frac{12}{14^{3/2}}, \frac{6}{14^{3/2}} > = 14^{-3/2} < 10, 12, 6 >. \]
Hence
\[ \mathbf{N}(-3/2) = \frac{\mathbf{T}'(-3/2)}{|\mathbf{T}'(-3/2)|} = \frac{<10, 12, 6>}{\sqrt{100 + 144 + 36}} = \frac{1}{\sqrt{280}} <10, 12, 6> = \frac{1}{\sqrt{70}} <5, 6, 3>. \]

And
\[ \mathbf{B}(-3/2) = \mathbf{T}(-3/2) \times \mathbf{N}(-3/2) = \frac{1}{14\sqrt{70}} < -3, 2, 1 > \times < 5, 6, 3 > = \]
\[ = \frac{1}{14\sqrt{5}} < 0, 14, -28 > = \frac{1}{\sqrt{5}} < 0, 1, -2 >. \]
(b) The curvature is equal to
\[
\frac{|\mathbf{T}'(-3/2)|}{|\mathbf{r}'(-3/2)|} = \frac{14^{-3/2}\sqrt{10^2 + 12^2 + 6^2}}{\sqrt{4(-3/2)^2 + 5}} = \frac{14^{-3/2}\sqrt{280}}{\sqrt{14}} = \frac{1}{14}\sqrt{\frac{5}{14}}.
\]

**Problem 37.** We have:
\[
\mathbf{r}'(t) = \langle 1, -\frac{1}{t^2}, 2t \rangle, \quad \mathbf{r}''(t) = \langle 0, \frac{2}{t^3}, 2 \rangle.
\]
So these vectors are orthogonal if and only if
\[
\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0, \quad -\frac{2}{t^5} + 4t = 0, \quad t^6 = \frac{1}{2}, \quad t = \pm \frac{1}{\sqrt{2}}.
\]
But \(t > 0\), so \(t = 1/\sqrt{2}\).

**Problem 38.** (a) \(\mathbf{r}'(t) = \langle 4 \cos t, 3, -4 \sin t \rangle\), \(|\mathbf{r}'(t)| = \sqrt{16 \cos^2 t + 9 + 16 \sin^2 t} = \sqrt{16 + 9} = 5\). Therefore,
\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \langle \frac{4}{5} \cos t, \frac{3}{5}, -\frac{4}{5} \sin t \rangle.
\]
(b)
\[
\mathbf{T}'(t) = \langle -\frac{4}{5} \sin t, 0, -\frac{4}{5} \cos t \rangle, \quad |\mathbf{T}'(t)| = \sqrt{\left(\frac{4}{5}\right)^2 \sin^2 t + \left(\frac{4}{5}\right)^2 \cos^2 t} = \frac{4}{5}.
\]
Therefore,
\[
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin t, 0, -\cos t \rangle.
\]
(c) This point corresponds to the value \(t = \pi/6\) (compare the second components: \(3t = \pi/2, \quad t = \pi/6\)). The tangent vector at this point is \(\mathbf{r}'(\pi/6) = \langle 2\sqrt{3}, 3, -2 \rangle\). This is a directional vector of this tangent line, which passes through the point \((2, \pi/2, 2\sqrt{3})\). The parametric equation of this tangent line is:
\[
x = 2 + 2\sqrt{3}t, \quad y = \pi/2 + 3t, \quad z = 2\sqrt{3} - 2t.
\]
(d) This plane passes through the point \((2, \pi/2, 2\sqrt{3})\) and has \(\mathbf{r}'(\pi/6)\) as a normal vector. So the equation of this plane is
\[
2\sqrt{3}(x - 2) + 3(y - \pi/2) - 2(z - 2\sqrt{3}) = 0.
\]

**Problem 39.** (b) Since \(r = 1 + \cos \theta\), we have: \(x = r \cos \theta = (1 + \cos \theta) \cos \theta, \quad y = r \sin \theta = (1 + \cos \theta) \sin \theta\). Denote \(\theta = t\). So we obtain the parametric equations of this curve:
\[
x = (1 + \cos t) \cos t, \quad y = (1 + \cos t) \sin t.
\]
At the point \(t\), the directional vector of the tangent line is \(< x'(t), y'(t) >\). But
\[
x'(t) = (- \sin t) \cos t + (1 + \cos t)(- \sin t) = - \sin t(1 + 2 \cos t),
\]
\[
y'(t) = (- \sin t) \sin t + (1 + \cos t) \cos t = \cos^2 t + \cos t - \sin^2 t.
\]
The tangent line is horizontal if and only if $y'(t) = 0$, i.e.
\[
\cos^2 t + \cos t - \sin^2 t = 0.
\]
Denote $\cos t = u$. Then $\sin^2 t = 1 - u^2$, and we have: $u^2 + u - (1 - u^2) = 0$, $2u^2 + u - 1 = 0$, $u = -1, 1/2$. Return to the initial variable: $\cos t = -1, 1/2, t = \pi/3, \pi, 5\pi/3, 7\pi/3, 3\pi$ (recall $t = \theta = [0, 3\pi]$).

The tangent line is vertical if and only if $x'(t) = 0$, i.e.
\[-\sin t(1 + 2 \cos t) = 0 \iff \sin t = 0 \text{ or } 1 + 2 \cos t = 0.
\]
But $\sin t = 0 \iff t = 0, \pi, 2\pi, 3\pi$; $1 + 2 \cos t = 0 \iff \cos t = -1/2 \iff t = 2\pi/3, 4\pi/3, 8\pi/3$.

**Problem 40.** Let $v(t)$ be the velocity, $a(t) = v'(t)$ be the acceleration. If $|v| = \text{const}$, then $v \cdot v = |v|^2 = \text{const}$, and differentiating this scalar product, we obtain (thanks to Theorem 3 from Section 12.3, Stewart) that
\[a \cdot v + v \cdot a = 0, \quad 2a \cdot v = 0, \quad a \cdot v = 0,
\]
hence $v$ and $a$ are orthogonal.

**Problem 41.** (a) $r'(t) = <1, 2t, -1/t^2>$, $r''(t) = <0, 2, 2/t^3>$. Hence $r'(1) = <1, 2, -1>$, $r''(1) = <0, 2, 2>$. By the formula from Section 12.3, Stewart, the vector projection of $r''(1)$ onto $r'(1)$ is
\[
\frac{r'(1) \cdot r''(1)}{|r'(1)|^2} r'(1) = \frac{1 \cdot 0 + 2 \cdot 2 + (-1) \cdot 2}{1^2 + 2^2 + (-1)^2} <1, 2, -1> = \frac{1}{3} <1, 2, -1>,
\]
and the scalar projection is the absolute value of the vector projection, i.e.
\[
\frac{1}{3} \sqrt{1^2 + 2^2 + (-1)^2} = \frac{\sqrt{6}}{3}.
\]
(b)
\[
r'(t) \perp r''(t) \iff r'(t) \cdot r''(t) = 0 \iff 1 \cdot 0 + 2t \cdot 2 + (-1) \cdot 2 = 0 \iff 4t - \frac{2}{t^3} = 0 \iff 4t^6 - 2 = 0 \iff t^6 = \frac{1}{2} \iff t = \pm \frac{1}{\sqrt[3]{2}}.
\]

**Problem 42.** In the three-dimensional space,
\[
r(t) = <t^2 - 3t, t^2 + 2t, 0>, \quad r'(t) = <2t - 3, 2t + 2, 0>, \quad r''(t) = <2, 2, 0>.
\]
Hence
\[
r'(t) \times r''(t) = <0, 0, 2(2t - 3) - 2(2t + 2) > = <0, 0, -10>;
\]
\[
|r'(t)| = \sqrt{(2t - 3)^2 + (2t + 2)^2} = \sqrt{8t^2 - 4t + 13}.
\]
Thus, the curvature at the point $t$ is
\[
\frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{10}{(8t^2 - 4t + 13)^{3/2}}.
\]
and it is maximal if and only if the denominator of this fraction is minimal, i.e. $8t^2 - 4t + 13$ is minimal. But $8t^2 - 4t + 13 = 8(t - 1/4)^2 - 1/2 + 13$ is minimal at the point $t = 1/4$. Hence the curvature is maximal at $t = 1/4$.

**Problem 43.** (b) Let us find the parametric equations of this curve:

$$x = r \cos \theta = 2 \cos \theta (1 - \cos \theta), \quad y = r \sin \theta = 2 \sin \theta (1 - \cos \theta).$$

The tangent line is horizontal if and only if the tangent vector $< x'(\theta), y'(\theta) >$ is horizontal, i.e. if and only if $y'(\theta) = 0$. But

$$y'(\theta) = 2 \cos \theta (1 - \cos \theta) + 2 \sin^2 \theta = 2 \cos \theta - 2 \cos^2 \theta + 2(1 - \cos^2 \theta) = 2 + 2u - 4u^2,$$

where $u = \cos \theta$. We solve the equation $2 + 2u - 4u^2 = 0$ and find $u = 1, -1/2$. But $\cos \theta = 1 \Leftrightarrow \theta = 0$, $\cos \theta = -1/2 \Leftrightarrow \theta = \pm 2\pi/3$. In the former case $r = 0$, so this is the origin. In the latter case $r = 2(1 - (-1/2)) = 3$. Thus, the answer: $r = 0; \quad r = 3, \theta = \pm 2\pi/3$. 

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3. Partial Derivatives. Chapter 14

Problem 44. (a) \(e^x/y \geq 0 \iff y > 0\), since for all \(x\) \(e^x > 0\); so the domain is \((x, y) : y > 0\).
(b) Note that \(f = e^{x/2}y^{-1/2}\). Hence \(f_x = (e^{x/2})(y^{-1/2}) = (1/2)e^{x/2}y^{-1/2}\),
\(f_y = e^{x/2}(y^{-1/2})' = e^{x/2}(-1/2)y^{-3/2} = -(1/2)e^{x/2}y^{-3/2}\),
\(f_{xy} = (1/2e^{x/2}y^{-1/2})y = 1/2e^{x/2}(y^{-1/2})y = (1/2)e^{x/2}(-1/2)y^{-3/2} = -(1/4)e^{x/2}y^{-3/2}\).
(c) \(z = 1 \iff y = e^x, z = 2 \iff y = e^{x/4}\).

Problem 45. (a) Let us find \(f_x(-2, 1), f_y(-2, 1)\). \(f_x = 2xy + 1\), hence \(f_x(-2, 1) = -3\).
\(f_y = x^2 + 3y^2\), hence \(f_y(-2, 1) = 7\). Thus, the equation of this plane is
\(z - 3 = -3(x + 2) + 7(y - 1), \ 3x - 7y + z + 10 = 0\).
(b) \(f_{xx} = 2y, f_{xy} = f_{yx} = 2x, f_{yy} = 6y\).

Problem 46. (a) \(f_x = 2x \sin(\pi y), f_y = \pi x^2 \cos(\pi y), f_{xy} = 2\pi x \cos(\pi y)\).
(b) First, note that \(f(3, 1) = 0\). Let us find \(f_x(3, 1), f_y(3, 1)\). \(f_x(3, 1) = 2 \cdot 3 \sin(\pi) = 0\),
\(f_y(3, 1) = \pi 3^2 \cos(\pi) = -9\pi\). Thus, the equation of this plane is
\(z - 0 = -0(x - 3) + (-9\pi)(y - 1), \ z + 9\pi y = 9\pi\).
(c) Any directional vector of this line is a normal vector to the tangent plane at the point
\((x, y) = (3, 1)\). E.g. we can take <0, 9\pi, 1> (indeed, look at the equation of this plane). Since
this line has the point \((3, 1, f(3, 1)) = (3, 1, 0)\), the equation of the line is \(x = 0t + 3 = 3, \ y = 9\pi t + 1, \ z = t\). (Of course, there may be equivalent equations, i.e. different answers that are still correct.)

Problem 47. Suppose the length is \(xf\), the width is \(yf\), the height is \(zf\). Then the volume
is \(xyzf^3\), hence \(xyz = 6\). The area of the bottom is \(xyft^2\), hence its cost is \(3xyf\$\). The total area
of the sides is \(2xz + 2yzft^2\) (since there are fours sides, two have area \(xzft^2\) and two have area
\(yzft^2\)). Hence its cost is \(2(2xz + 2yz)f\$\). From now on, we will eliminate the dollar sign for the sake of brevity. The total cost is
\(f := 3xy + 4xz + 4yz\).

But \(z = 6/(xy)\), hence \(f\) can be expressed as a function of two variables \(x, y\):
\(f(x, y) = 3xy + \frac{24}{x} + \frac{24}{y}\).

Let us find its critical points:
\(f_x = 3y - \frac{24}{x^2} = 0, \ f_y = 3x - \frac{24}{y^2} = 0\).

We need to solve this system of equations. After simple algebraic operations, we obtain:
\(x^2y = 8, \ xy^2 = 8\).

Divide the first equation by the second; obtain: \(x/y = 1, \ x = y\). Hence \(x^3 = 8, \ x = 2, \ y = 2\). You
do not need to verify that this is indeed a maximal point, on the midterm, unless it is required explicitly (as in this case).
\(f_{xx} = \frac{48}{y^3}, \ f_{xy} = f_{yx} = 3, \ f_{yy} = \frac{48}{x^3}\).
Plug in \( x = y = 2 \):

\[
f_{xx}(2, 2) = \frac{48}{8} = 6, \quad f_{xy}(2, 2) = f_{yx}(2, 2) = 3, \quad f_{yy}(2, 2) = \frac{48}{8} = 6.
\]

Now we apply the Second Derivative Test: since \( f_{xx}(2, 2) > 0 \) and \( f_{xx}(2, 2)f_{yy}(2, 2) - f_{xy}(2, 2)^2 = 6 \cdot 6 - 3^2 = 27 > 0 \), we see: \((2, 2)\) is indeed a local minimum point. And \( z = 6/(xy) = 6/4 = 3/2 \).

The answer: The length and the width of the bottom are 2 ft each, the height is 1.5 = 3/2 ft.

**Problem 48.** \( f(2, 2) = \sqrt{28 - 8 - 4} = \sqrt{16} = 4 \).

\[
f_x = \frac{(28 - 2x^2 - y^2)_x}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-4x}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-2x}{\sqrt{28 - 2x^2 - y^2}},
\]

\[
f_y = \frac{(28 - 2x^2 - y^2)_y}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-2y}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-y}{\sqrt{28 - 2x^2 - y^2}}.
\]

Plug in \( x = y = 2 \): \( \sqrt{28 - 2x^2 - y^2} = 4 \), hence

\[
f_x(2, 2) = \frac{-4}{4} = -1, \quad f_y(2, 2) = \frac{-2}{4} = -\frac{1}{2}.
\]

Thus, the tangent plane is

\[
z - f(2, 2) = f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2),
\]

or, in other words

\[
z - 4 = (-1)(x - 2) + \left(-\frac{1}{2}\right)(y - 2) = -x - \frac{y}{2} + 3, \quad z = -x - \frac{y}{2} + 7.
\]

This equation can be considered as a linear approximation of the function \( f \) in the neighborhood of \((2, 2)\). For example, \( f(1.95, 2.01) \approx -1.95 - 2.01/2 + 7 = -2 + 0.05 - 1 - 0.005 + 7 = 4 + 0.045 = 4.045 \).

**Problem 49.** It is obvious that \( f(x, y) \geq 0 \) for any point \((x, y) \in D\). And \( f = 0 \iff x = 0 \) or \( y = 0 \). Hence \( f_{\text{min}} = 0 \), attained at any point with \( x = 0 \) or \( y = 0 \) on the boundary. It is much more difficult to find \( f_{\text{max}} \). First, let us find the critical points inside \( D \):

\[
f_x = 3y^2 = 0, \quad f_y = 6xy = 0;
\]

but this implies \( y = 0 \), i.e. there is no critical point inside \( D \) (only on the boundary, and we cannot take them into account if we try to find the maximal value in \( D \)). Hence \( f \) does not attain its maximal value in \( D \) in the interior of \( D \). It attains this maximal value in \( D \) on the boundary of \( D \). But what is this value? The boundary consists of three pieces:

\[
x = 0, \quad y \in [0, 3]; \quad x \in [0, 3], \quad y = 0; \quad x^2 + y^2 = 9, \quad x, y \geq 0 \ (\Rightarrow x, y \leq 3).
\]

On the first and second pieces we have \( f = 0 \). And on the third piece

\[
f = 3x(9 - x^2) = 27x - 3x^3, \quad x \in [0, 3].
\]

Let us find its maximal value on this interval (now we temporarily consider \( f \) as a function of one variable).

\[
f_x = 27 - 9x^2 = 0, \quad x^2 = 3, \quad x = \sqrt{3}.
\]

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And $f_{xx} = -18x < 0$ for $x = \sqrt{3}$. Hence the Second Derivative Test shows that $\sqrt{3}$ is a point of local maximum.

Since this is the only critical point, it is the point of global maximum on $[0, 3]$. Thus $(x, y) = (\sqrt{3}, \sqrt{6})$ (we find the corresponding value of $y$ in this way: $y = \sqrt{9 - x^2} = \sqrt{9 - 3} = \sqrt{6}$) is a maximal point of $f$ at the third piece of the boundary.

We conclude that this is the point of global maximum of $f$ on $D$, since its values on the other two pieces of the boundary are $0 < f(\sqrt{3}, \sqrt{6}) = 18\sqrt{3}$. Thus $18\sqrt{3} = f_{max}$. Summarizing these results, we have:

\[ f_{min} = 0, \text{ attained at any point } x = 0 \text{ or } y = 0. \]
\[ f_{max} = 18\sqrt{3}, \text{ attained at } (\sqrt{3}, \sqrt{6}). \]

**Problem 50.** (a) The domain of $f$ is given by the conditions

\[ \ln(2x - y) \neq 0 \iff 2x - y > 0, \quad 2x - y \neq 1 \iff y < 2x, \quad y \neq 2x - 1. \]

(b) $z_0 = f(x_0, y_0) = 3e^2/\ln e = 3e^2$.

\[
\begin{align*}
  f_x(x, y) &= \frac{4x \ln(2x - y) - (2x^2 + y^2) \frac{2}{2x-y}}{\ln^2(2x - y)}, \\
  f_y(x, y) &= \frac{2y \ln(2x - y) - (2x^2 + y^2) \frac{-1}{2x-y}}{\ln^2(2x - y)},
\end{align*}
\]

and at the point $(e, e)$ we have

\[
\begin{align*}
  f_x(e, e) &= \frac{4e \ln e - 3e^2}{\ln^2 e} = -2e, \\
  f_y(e, e) &= \frac{2e - 3e^2}{\ln^2 e} = 5e.
\end{align*}
\]

So the equation of the tangent plane is

\[ z - z_0 = f_x(e, e)(x - e) + f_y(e, e)(y - e), \quad z - 3e^2 = -2e(x - e) + 5e(y - e) = -2ex + 5ey - 3e^2, \]

or

\[ z = -2ex + 5ey. \]

(c) $f(3, 3) \approx -2e \cdot 3 + 5e \cdot 3 = 9e$.

**Problem 51.** $F(1, 1) = 1/12$;

\[
\begin{align*}
  F_x(x, y) &= -\frac{5(3y - 2)}{(5x + 7)^2}, \\
  F_y(x, y) &= \frac{3}{5x + 7}.
\end{align*}
\]

So $F_x(1, 1) = -5/144, \quad F_y(1, 1) = 1/4$. The equation of the tangent plane is

\[ z - \frac{1}{12} = -\frac{5}{144}(x - 1) + \frac{1}{4}(y - 1). \]

**Problem 52.** Since $x + y + z = 100$, we have $x = 100 - y - z$. And the expression which we need to maximize is

\[ f(y, z) = (100 - y - z)y^2z^3 = 100y^2z^3 - y^3z^3 - y^2z^4. \]

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Take the derivatives:

\[ f_y = 200yz^3 - 3y^2z^3 - 2yz^4, \quad f_z = 300y^2z^2 - 3y^3z^3 - 4y^2z^3. \]

The system of equations \( f_y = 0, \quad f_z = 0 \) can be rewritten as

\[ 3y + 2z = 200, \quad 3y + 4z = 300. \]

Subtract the first equation from the second one and get: \( 2z = 100; \quad z = 50; \quad y = 100/3; \quad x = 100 - y - z = 100/6. \) The answer: \( x = 100/6, \quad y = 100/3, \quad z = 50. \)

**Problem 53.** First, let us solve the system of equations \( f_x = 0, \quad f_y = 0. \)

\[ f_x = 3x^2 - 12, \quad f_y = -6 + 2y. \]

Therefore, \( f_x = f_y = 0 \iff x^2 = 4, \quad y = 3 \iff x = \pm 2, \quad y = 3. \)

Also, \( f_{xx} = 6x, \quad f_{xy} = 0, \quad f_{yy} = 2, \) so \( D := f_{xx}f_{yy} - f_{xy}^2 = 12x. \) The point \( (2, 3) \) satisfies the conditions \( f_{xx} > 0, \quad D > 0, \) so this is a local minimum point. The point \( (-2, 3) \) satisfies the condition \( D < 0, \) so it is a saddle point.

**Problem 54.** Suppose \( x \) is the width of the glass side, \( y \) is the width of concrete sides, \( z \) is the height. (Everything is measured in meters.) Then \( xyz \) is the volume of the pool, so \( xyz = 1000, \quad z = 1000/xy. \) The area of the glass side is \( xz, \) its cost is \( 100xz. \) The area of each of the adjacent (concrete) sides is \( yz, \) so its cost is \( 15yz. \) The area of the opposite (concrete) side is \( xz, \) so its cost is \( 15xz. \) The area of the concrete bottom is \( xy, \) its cost is \( 15xy. \) The total cost is

\[ 115xz + 30yz + 15xy. \]

We can express it as a function of \( x, y: \)

\[ f(x, y) = 115x \frac{1000}{xy} + 30y \frac{1000}{xy} + 15xy = \frac{115000}{y} + \frac{30000}{x} + 15xy. \]

Let us find its minimum points. Introduce a system of equations:

\[ f_x = -\frac{30000}{x^2} + 15y = 0, \quad f_y = -\frac{115000}{y^2} + 15x = 0, \]

so \( y = 2000/x^2 \) (from the first equation). Plug it into the second equation:

\[ -\frac{115000}{(2000/x^2)^2} + 15x = 0 \iff \frac{115000}{4000000}x^4 = 15x \iff x^3 = \frac{60000}{115} = \frac{20000}{23}, \]

so

\[ x = \left( \frac{20000}{23} \right)^{1/3}, \quad y = \frac{2000}{x^2} = \frac{2000}{(20000/23)^{1/3}}, \quad z = \frac{1000}{xy} = \frac{1}{2} \left( \frac{20000}{23} \right)^{1/3}. \]
4. Double Integrals. Chapter 15

Problem 55. The area of $D$ is
\[ \int \int_D 1 \, dx \, dy. \]

By Fubini’s Theorem, this integral equals
\[ \int_1^2 \left[ \int_{\ln x} e^x \, dy \right] \, dx = \int_1^2 (e^x - \ln x) \, dx. \]

But
\[ \int_1^2 e^x \, dx = e^x \bigg|_{x=1}^{x=2} = e^2 - e. \]

How to compute the integral of $\ln x$? We need to find the antiderivative of this function. One common mistake was that
\[ \int_1^2 \ln x \, dx = \frac{1}{x} \bigg|_{x=1}^{x=2}. \]

But this is not true, since $(1/x)' \neq \ln x$! To find the antiderivative of $\ln x$, we need to integrate by parts:
\[ \int \ln x \, dx = \int x' \ln x \, dx = x \ln x - \int x(\ln x)' \, dx = x \ln x - \int \frac{1}{x} \, dx = x \ln x - \int \frac{dx}{x} = x \ln x - \ln x. \]

Hence $(x \ln x - x)' = \ln x$, and by the Fundamental Theorem of Calculus
\[ \int_1^2 \ln x \, dx = (x \ln x - x) \bigg|_{x=1}^{x=2} = (2 \ln 2 - 2) - (\ln 1 - 1) = 2 \ln 2 - 1 = \ln 4 - 1. \]

Hence the area of $D$ is
\[ e^2 - e - 2 \ln 2 - 1. \]

Problem 56. First, find this one-variable integral:
\[ \int_0^1 xy \sin(x^2 y) \, dx = \int_0^y \sin \left( \frac{t}{2} \right) \, dt, \]

where we have changed the variables: $t = x^2 y$, $0 \leq t \leq y$, $dt = 2xy \, dx$. And
\[ \int_0^y \sin t \, dt = \frac{1}{2} (-\cos t) \bigg|_{t=0}^{t=y} = \frac{1}{2} (1 - \cos y). \]

Hence by Fubini’s Theorem
\[ \int_0^\pi/2 \int_0^1 xy \sin(x^2 y) \, dx \, dy = \int_0^{\pi/2} \left[ \int_0^1 xy \sin(x^2 y) \, dx \right] \, dy = \int_0^{\pi/2} \frac{1}{2} (1 - \cos y) \, dy = \]
\[
\frac{1}{2} \int_{0}^{\pi/2} dy - \frac{1}{2} \int_{0}^{\pi/2} \cos y dy = \frac{1}{2} \pi - \frac{1}{2} (\sin y) \bigg|_{y=\pi/2}^{y=0} = \frac{\pi}{4} - \frac{1}{2}.
\]

**Problem 57.** Since \((ye^{xy})_x = y^2e^{xy}\), we have by the Fundamental Theorem of Calculus:
\[
\int_{0}^{y} y^2e^{xy} dx = ye^{xy} \bigg|_{x=0}^{x=y} = ye^{y^2} - y.
\]

Hence by Fubini’s Theorem,
\[
\iint_{D} y^2e^{xy} dxdy = \int_{0}^{3} \left( \int_{0}^{y} y^2e^{xy} dx \right) dy = \int_{0}^{3} (ye^{y^2} - y) dy = \int_{0}^{3} ye^{y^2} dy - \int_{0}^{3} y dy.
\]

To find the first integral, change variables: \(t = y^2\), \(0 \leq t \leq 9\), \(dt = 2y dy\).
\[
\int_{0}^{3} ye^{y^2} dy = \int_{0}^{9} e^{t} \frac{dt}{2} = \frac{e^{9} - 1}{2}.
\]

And the second integral is much easier to compute:
\[
\int_{0}^{3} y dy = \frac{y^2}{2} \bigg|_{y=0}^{y=3} = \frac{9}{2}.
\]

Thus, the answer is
\[
\frac{1}{2}(e^{9} - 1) - \frac{9}{2} = \frac{e^{9}}{2} - 5.
\]

**Problem 58.** This body is between \(z = 0\) and \(z = 12 - 3x - 2y\) and above the rectangle \(R\). So its volume is
\[
\iint_{R} (12 - 3x - 2y) dA = \int_{2}^{3} \int_{0}^{1} (12 - 3x - 2y) dxdy.
\]

But
\[
\int_{0}^{1} (12 - 3x - 2y) dx = \left( 12x - \frac{3x^2}{2} - 2xy \right) \bigg|_{x=0}^{x=1} = 12 - \frac{3}{2} - 2y = \frac{21}{2} - 2y.
\]

Therefore,
\[
\int_{2}^{3} \int_{0}^{1} (12 - 3x - 2y) dxdy = \int_{2}^{3} \left( \frac{21}{2} - 2y \right) dy = \left( \frac{21}{2} y - y^2 \right) \bigg|_{y=2}^{y=3} = \frac{21}{2} - 5 = \frac{11}{2}.
\]

**Problem 59.** Let us change the order of integration. \(y \leq 4 - x^2 \iff x^2 \leq 4 - y \iff x \leq \sqrt{4-y} \). Therefore, the domain of integration is \(0 \leq y \leq 4\), \(0 \leq x \leq \sqrt{4-y}\). And the integral is equal to
\[
\int_{0}^{4} \left( \int_{0}^{\sqrt{4-y}} \frac{e^{2y}}{4-y} dx \right) dy = \int_{0}^{4} \frac{x}{2} \bigg|_{x=0}^{x=\sqrt{4-y}} \frac{e^{2y}}{4-y} dy = \int_{0}^{4} \frac{4-y}{2} \frac{e^{2y}}{4-y} dy = \frac{e^{4}}{2} - 5.
\]
\[
\frac{1}{2} \int_0^4 e^{2y} dy = \left. \frac{1}{2} e^{2y} \right|_{y=0}^{y=4} = e^8 - \frac{1}{4}.
\]

**Problem 60.** If \(D = \{0 \leq r \leq 1 + \cos \theta, \ 0 \leq \theta \leq \pi\}\) is this region, then its area is

\[
\iint_D 1 \, dxdy = \int_0^\pi \int_0^{1+\cos \theta} r \, dr \, d\theta = \frac{1}{2} \int_0^\pi (1 + \cos \theta)^2 \, d\theta.
\]

But

\[
\frac{1}{2} (1 + \cos \theta)^2 = \frac{1}{2} (1 + 2 \cos \theta + \cos^2 \theta) = \frac{1}{2} (1 + 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta)) = \frac{3}{4} + \cos \theta + \frac{1}{4} \cos 2\theta.
\]

Since

\[
\int_0^\pi \cos \theta \, d\theta = \sin \theta \big|_{\theta=\pi} = 0, \quad \int_0^\pi \cos 2\theta \, d\theta = \frac{1}{2} \sin 2\theta \big|_{\theta=\pi} = 0,
\]
we have:

\[
\iint_D 1 \, dxdy = \int_0^\pi \frac{3}{4} \, d\theta = \frac{3}{4} \pi.
\]

**Problem 61.** (b) In polar coordinates, \(x^2 + y^2 = r^2, \ x = r \cos \theta,\) and \(D = \{4 \leq r \leq 4 \cos \theta, \ 0 \leq \theta \leq \pi\}\) = \(\{2 \leq r \leq 4 \cos \theta, \ 0 \leq \theta \leq \pi\}\) = \(\{2 \leq r \leq 4 \cos \theta, \ \cos \theta \geq 1/2, \ 0 \leq \theta \leq \pi\}\)

\[
= \{2 \leq r \leq 4 \cos \theta, \ 0 \leq \theta \leq \pi/3\}.
\]

The area of this domain is

\[
\iint_D 1 \, dxdy = \int_0^{\pi/3} \int_0^{4 \cos \theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{\pi/3} 4 \cos \theta \, d\theta = \frac{1}{2} \int_0^{\pi/3} (16 \cos^2 \theta - 4) \, d\theta.
\]

But \((16 \cos^2 \theta - 4)/2 = 8 \cos^2 \theta - 2 = 4(1 + \cos 2\theta) - 2 = 2 + 4 \cos 2\theta.\) Therefore, this integral is equal to

\[
(2 \sin 2\theta + 2\theta) \bigg|_{\theta=\pi/3} = 2 \sin(2\pi/3) - 2 \sin 0 + \frac{2\pi}{3} = \sqrt{3} + \frac{2\pi}{3}.
\]

**Problem 62.** (b) In polar coordinates, \(D = \{4 \leq r \leq 5, \ 0 \leq \theta \leq \pi/2\}.\) Also, \(\sqrt{x^2 + y^2} = r, \ x = r \cos \theta.\) Let us rewrite this integral in polar coordinates:

\[
\frac{\pi}{2} \int_0^5 (r \cos \theta + r) \, r \, dr \, d\theta = \frac{\pi}{2} \int_0^5 (1 + \cos \theta) \, r \, dr \, d\theta = \frac{\pi}{4} \int_4^5 r^3 \, dr = \frac{\pi}{2} \left( \frac{125}{3} - \frac{64}{3} \right) = \frac{61}{6} \pi.
\]

**Problem 63.** The area \(A\) of this region \(R = \{0 \leq \theta \leq \pi, \ 1 \leq r \leq 2\}\) is the difference of the area of two semi-discs, one with radius 2, the other with radius 1. The area of a semi-disc
with radius 1 is \(1^2 \pi/2 = \pi/2\); the area of a semi-disc with radius 2 is \(2^2 \pi/2 = 2\pi\). The difference between them is \(3\pi/2\). So \(A = 3\pi/2\).

The integral \(I\) of this function \(e^{-x^2-y^2} = e^{-r^2}\) over \(R\) can be rewritten as

\[
\int_0^\pi \int_0^2 e^{-r^2} r \, dr \, d\theta.
\]

But (after changing variables \(u = r^2, \, du = 2r \, dr\))

\[
\int_1^2 e^{-r^2} r \, dr = \int_1^4 e^{-u} \frac{du}{2} = \frac{1}{2} (-e^{-u}) \bigg|_{u=1}^{u=2} = \frac{1}{2} (e^{-1} - e^{-4}).
\]

Therefore,

\[
I = \int_0^\pi \int_0^2 e^{-r^2} r \, dr \, d\theta = 2\pi \frac{1}{2} (e^{-1} - e^{-4}) = \pi (e^{-1} - e^{-4}).
\]

And the average value is \(I/A = 2/3(e^{-1} - e^{-4})\).

**Problem 64.** The upper limit for \(r\) is always \(2(1+\cos \theta)\). The lower limit is \(2 \cos \theta\), if \(\cos \theta > 0\) (i.e. if \(-\pi/2 < \theta < \pi/2\)), and 0 otherwise (i.e. if \(\pi/2 \leq \theta \leq 3\pi/2\)). So this area is

\[
\int_{-\pi/2}^{\pi/2} \int_{2 \cos \theta}^{2(1+\cos \theta)} r \, dr \, d\theta + \int_{\pi/2}^{3\pi/2} \int_0^{2(1+\cos \theta)} r \, dr \, d\theta.
\]

Let us compute the first iterated integral. The inner integral is

\[
\int_{2 \cos \theta}^{2(1+\cos \theta)} r^2 \, dr = \frac{1}{2} \left(4(1 + \cos \theta)^2 - 4 \cos^2 \theta \right) = 2(1 + 2 \cos \theta) = 2 + 4 \cos \theta.
\]

Therefore, the double integral is

\[
\int_{-\pi/2}^{\pi/2} (2 + 4 \cos \theta) \, d\theta = 12 + 4 \sin \theta \bigg|_{-\pi/2}^{\pi/2} = 2\pi + 8.
\]

Let us compute the second iterated integral. The inner integral is

\[
\int_0^{2(1+\cos \theta)} r \, dr = \frac{(2(1 + \cos \theta))^2}{2} = 2(1 + \cos \theta)^2 =
\]

\[
= 2 \left(1 + 2 \cos \theta + \cos^2 \theta \right) = 2 + 4 \cos \theta + 1 + \cos 2\theta = 3 + 4 \cos \theta + \cos 2\theta.
\]

So the double integral is

\[
\int_{\pi/2}^{3\pi/2} (3 + 4 \cos \theta + \cos 2\theta) \, d\theta = 3\theta + 4 \sin \theta + \frac{1}{2} \sin 2\theta \bigg|_{\pi/2}^{3\pi/2} = 3\pi - 8.
\]
Sum them up: \[(2\pi + 8) + (3\pi - 8) = 5\pi.\]

This is the answer.

**Problem 65.** First, let us calculate the mass \(m(a)\) of the lamina for every fixed value of the parameter \(a\).

\[
m(a) = \int_1^2 \int_{ax}^{2ax} \rho(x, y) dy \, dx = \int_1^2 \int_{ax}^{2ax} \left(\frac{1}{x} + \frac{1}{y^2}\right) dy \, dx = \int_1^2 \left(\frac{y}{x} - \frac{1}{y}\right) \bigg|_{y=ax}^{2} \, dx =
\]

\[
\int_1^2 \left(\frac{2ax}{x} - \frac{ax}{x} - \frac{1}{2ax} + \frac{1}{ax}\right) \, dx = \int_1^2 \left(a + \frac{1}{2ax}\right) \, dx =
\]

\[
\left(\frac{ax}{2} + \frac{1}{2} \log x\right) \bigg|_{x=1}^{x=2} = a + \frac{\log 2}{2a}.
\]

When does this function attains its minimum? Use differential calculus of one variable:

\[
m'(a) = 1 - \frac{\log 2}{2} \frac{1}{a^2} = 0 \Rightarrow a^2 = \frac{\log 2}{2} \Rightarrow a = \sqrt{\frac{\log 2}{2}}.
\]

**Problem 66.** Suppose \(m\) is the center of mass of the lamina, \((x_c, y_c)\) is its center of mass. By symmetry, it lies on the \(y\)-axis, which is equivalent to \(x_c = 0\), and it suffices to find \(y_c\).

\[
y_c = \frac{1}{M} \int_D y \rho(x, y) \, dx \, dy, \quad M = \int_D \rho(x, y) \, dx \, dy,
\]

where \(D\) is the region occupied by lamina. Let us convert these integrals to polar coordinates.

\[
D = \{1 < r < 5, \ 0 < \theta < \pi\},
\]

because the first semicircle has radius 1, the second one has radius \(\sqrt{25} = 5\), and \(D\) lies above the \(x\)-axis (this implies restrictions on \(\theta\)). Moreover,

\[
\rho = \frac{k}{r}, \quad dA = rdr\theta, \quad y = r \sin \theta.
\]

Therefore,

\[
M = \int_0^\pi \int_1^5 \frac{k}{r} rdrd\theta = k \int_1^5 dr \int_0^\pi d\theta = 4k\pi,
\]

\[
\int_D y \rho(x, y) \, dx \, dy = \int_1^\pi \int_0^r \sin \theta \frac{k}{r} rdrd\theta = k \int_0^\pi \sin \theta d\theta \int_0^5 r \, dr = 24k
\]

(after calculating these integrals). Thus,

\[
y_c = \frac{24k}{4k\pi} = \frac{6}{\pi}.
\]
5. Taylor Polynomials and Series

Problem 67. (a) First, let us calculate the first two derivatives and the function itself at the point \( x = b = 1 \).

\[
f(x) = (3 + x)^{1/2} \Rightarrow f(1) = 2;
\]
\[
f'(x) = \frac{1}{2}(3 + x)^{-1/2} \Rightarrow f'(1) = \frac{1}{4};
\]
\[
f''(x) = \frac{1}{2} \left( -\frac{1}{2} \right) (3 + x)^{-3/2} = -\frac{1}{4}(3 + x)^{-3/2} \Rightarrow f''(1) = -\frac{1}{32}.
\]
Thus,
\[
T_2(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2 = 2 + \frac{1}{4}(x - 1) - \frac{1}{64}(x - 1)^2.
\]

(b) \( \sqrt{3.7} = \sqrt{3 + 0.7} = f(0.7) \approx T_2(0.7) = 2 - \frac{0.3}{4} - \frac{0.9}{64} \).

(c) The Taylor inequality gives us the following upper estimate:
\[
\frac{M}{3!} |0.7 - 1|^3, \quad M := \max_{x \in [0.7,1]} |f^{(3)}(x)|.
\]
Let us find \( M \):
\[
f^{(3)}(x) = -\frac{1}{4} \left( -\frac{3}{2} \right) (3 + x)^{-5/2} = \frac{3}{8}(3 + x)^{-5/2}, \quad |f^{(3)}(x)| = \frac{3}{8}(3 + x)^{-5/2},
\]
and this function decreases on \([0.7,1]\), hence it attains its maximum at the left endpoint of this interval:
\[
M = |f^{(3)}(0.7)| = \frac{3}{8}(3.7)^{-5/2}.
\]
Thus, the upper bound for this error is
\[
\frac{3}{8}(3.7)^{-5/2} \frac{0.3^3}{6} = \frac{(0.3)^3}{16(3.7)^{5/2}}.
\]
Note: You may leave the answers in this exact form. Do not waste your time on the final exam using a calculator to find the decimal approximation.

Problem 68. (a) Similarly, we find \( f(1) = 2, f'(1) = (3x^2 + 1)|_{x=1} = 4, f''(1) = (6x)|_{x=1} = 6 \), and \( T_2(x) = 2 + 4(x - 1) + 6(x - 1)^2/2 = 2 + 4(x - 1) + 3(x - 1)^2 \).

(b) Let \( J = [1 - \varepsilon, 1 + \varepsilon], \varepsilon > 0 \) is unknown, the question of the problem is to find it. The error bound is
\[
\frac{M}{3!}\varepsilon^3, \quad M := \max_{x \in [1-\varepsilon,1+\varepsilon]} |f^{(3)}(x)|.
\]
But \( f^{(3)}(x) = 6 \), this is a constant function, hence \( M = 6 \), and the error bound is simply \( \varepsilon^3 \). And \( \varepsilon^3 < 0.001 = (0.1)^3 \Leftrightarrow \varepsilon < 0.1 \), so you may take any \( \varepsilon < 0.1 \), e.g. 0.05 or 0.09.

Problem 69. (a) \( f'(x) = 1/(5 - x)^2 \), so \( f(0) = 1/5, f''(0) = 1/25, T_1(x) = \frac{1}{5} + \frac{x}{25} \).
(b) \( f''(x) = 2/(5-x)^3 \), \( |f''(x)| = 2/(5-x)^3 \), this function increases on \( I \), since the denominator \((5-x)^3 \) decreases on \( I \); therefore, it attains its maximum on this interval at its right endpoint,

\[
M := \max_{x \in I} |f''(x)| = |f''(2)| = \frac{2}{27}.
\]

The upper bound is \((M/2) \cdot 2^2 = 4/27\).

(c) It can be shown that for \( k = 1, 2, 3, \ldots \) we have

\[
f^{(k)}(x) = \frac{k!}{(5-x)^{k+1}}.
\]

Therefore,

\[
M_k := \max_{x \in I} |f^{(k)}(x)| = k! \max_{x \in I} \frac{1}{(5-x)^{k+1}} = \frac{1}{(5-2)^{k+1}} = \frac{k!}{3^{k+1}}.
\]

The upper bound for \(|f(x) - T_n(x)|\) is

\[
B_n := \frac{M_{n+1}}{(n+1)!} \cdot 2^{n+1} = \frac{(n+1)!}{3^{n+2}(n+1)!} \cdot 2^{n+1} = \frac{1}{3} \left( \frac{2}{3} \right)^{n+1}.
\]

We must find all \( n \) such that \( 0.04 < B_n < 0.05 \). The sequence \( B_1, B_2, B_3, \ldots \) decreases. \( B_1 = \frac{4}{27} > 0.05, B_2 = \frac{8}{81} > 0.05, B_3 = \frac{16}{243} > 0.05, B_4 = \frac{32}{729} \in (0.04, 0.05), B_5 = \frac{64}{2187} < 0.04 \). Thus, the only \( n \) such that \( 0.04 < B_n < 0.05 \) is \( n = 4 \).

**Problem 70.** By the Taylor inequality, the upper bound is

\[
B_n := \frac{M_{n+1}}{(n+1)!} \cdot (0.5)^{n+1}, \quad M_k := \max_{x \in [-0.5, 0]} |g^{(k)}(x)|.
\]

But \( g^{(k)}(x) = 2^k e^{2x} \), \( |g^{(k)}(x)| = 2^k e^{2x} \), this function increases, hence this maximum is attained at the right endpoint: \( M_k = |g^{(k)}(0)| = 2^k \). Therefore,

\[
B_n = \frac{1}{(n+1)!}.
\]

The sequence \( B_1, B_2, \ldots \) decreases. \( B_3 = 1/24 > 0.01, B_4 = 1/120 < 0.01 \). Hence we can take \( n = 4 \).

Note: Of course, we can take a larger \( n \), but this is not advisable, since it would be harder to compute the Taylor approximation. But if you write e.g. \( n = 6 \), this would be a correct answer.

**Problem 71.** (a) Let us find the values of this function and its first and second derivatives at \( b = 4 \):

\[
f(x) = \sin(x - 4) + \cos(x - 4) + 4\sqrt{x} \Rightarrow f(4) = 9;
\]

\[
f'(x) = \cos(x - 4) - \sin(x - 4) + 2x^{-1/2} \Rightarrow f'(4) = 2;
\]

\[
f''(x) = -\sin(x - 4) - \cos(x - 4) - x^{-3/2} \Rightarrow f''(4) = -1 - \frac{1}{8} = -\frac{9}{8}.
\]

Therefore,

\[
T_2(x) = 9 + 2(x - 4) - \frac{9}{16}(x - 4)^2.
\]
(b) \(f(4.1) \approx T_2(4.1) = 9 + 0.2 - \frac{0.09}{16}.

(c) The upper bound is \((M/3!)(4.1 - 4)^3\), \(M := \max_{x \in [4,4.1]} |f^{(3)}(x)|\). For \(x \in [4,4.1]\),

\[
f^{(3)}(x) = -\cos(x - 4) + \sin(x - 4) + \frac{3}{2}x^{-5/2},
\]

\[
|f^{(3)}(x)| \leq |\cos(x - 4)| + |\sin(x - 4)| + \frac{3}{2}x^{-5/2} \leq 1 + 1 + \frac{3}{2} \cdot \frac{4^{-5/2}}{4} = 2 + \frac{3}{64} < 3.
\]

Hence, \(M < 3\), and we may take the upper bound to be \((3/3!)(0.1)^3 = 0.0005\). (In fact, it is much smaller.)

**Problem 72.** (a) \(f(x) = x \ln x \Rightarrow f(1) = 0\); \(f'(x) = 1 + \ln x \Rightarrow f'(1) = 1\); \(f''(x) = 1/x \Rightarrow f''(1) = 1\). So

\[
T_2(x) = (x - 1) + \frac{(x - 1)^2}{2}.
\]

(b) \(f'''(x) = -1/x^2\), and \(|f'''(x)| = 1/x^2\) decreases, so if we take \(J = [1 - \varepsilon, 1 + \varepsilon]\) then

\[
M_3 := \max_J |f'''(x)| = \frac{1}{(1 - \varepsilon)^2}
\]

is attained at the left endpoint. The Quadratic Approximation Error Bound is

\[
\frac{M}{3!} \varepsilon^3 = \frac{\varepsilon^3}{6(1 - \varepsilon)^2}.
\]

Take, e.g. \(\varepsilon = 0.2\). Then this Error Bound is \(0.008/(6(0.8)^2) < 0.01\) (just calculate).

**Problem 73.** (a)

\[
\ln(e + 3x) = \ln e + \ln(1 + 3x/e) = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(3x/e)^n}{n} = 1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(3/e)^n}{n} x^n.
\]

(b) Since \(\ln(1 + t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n}\) converges for \(-1 < t < 1\), the series from (a) converges for

\[
-1 < \frac{3x}{e} < 1 \Leftrightarrow -\frac{e}{3} < x < \frac{e}{3}.
\]

**Problem 74.** (a)

\[
f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (3x^2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n} x^{2n}.
\]

(b) Similarly to Problem 2b, this series converges if

\[
-1 < 3x^2 < 1 \Leftrightarrow x^2 < 1/3 \Leftrightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}.
\]

**Problem 75.**

\[
\frac{1}{x} (e^{x^2} - 1) = \frac{1}{x} \left( \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} - 1 \right) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!};
\]

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\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n, \quad \frac{1}{(1-x)^2} = \left( \frac{1}{1-x} \right)' = \sum_{n=1}^{\infty} nx^{n-1}.
\]

Thus,
\[
f(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!} - 3 \sum_{n=1}^{\infty} nx^{n-1}.
\]

Let us expand it out to find the first four terms:
\[
f(x) = \left( x - \frac{x^3}{2} + \frac{x^5}{24} - \ldots \right) - 3 \left( 1 + 2x + 3x^2 + 4x^3 \right) = -3 - 5x - 9x^2 - \frac{25}{2} x^3 + \ldots,
\]

where by \ldots we denote the higher-order terms. Thus, the first four terms are
\[
-3 - 5x - 9x^2 - \frac{25}{2} x^3.
\]

\textbf{Problem 76. (a)}

\[
\frac{1}{2x^2 + 1} = \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-2)^n x^{2n},
\]

\[
\cos(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (3x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} x^{2n}.
\]

Thus
\[
f(x) = \sum_{n=0}^{\infty} \left( (-2)^n - \frac{(-1)^n 3^{2n}}{(2n)!} \right) x^{2n}.
\]

(b) The series for \( \cos 3x \) converges everywhere, and the series for \( 1/(1 + 2x^2) \) converges for \(-1 < 2x^2 < 1 \iff x^2 < 1/2 \iff -1/\sqrt{2} < x < 1/\sqrt{2} \). So this is the interval of convergence for the Taylor series for \( f(x) \).

(c) There are two ways to find \( T_n(x) \) - the \( n \)-th Taylor polynomial of \( f \) centered at \( x = b \). First: compute \( f(b), f'(b), f''(b), \ldots, f^{(n)}(b) \) and write
\[
T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!} (x - b)^k.
\]

Another way is to find Taylor series for \( f \) based at \( x = b \) and then take the the terms that contain \( (x - b)^k \), \( k = 0, 1, \ldots, n \), i.e. the terms of degree \( \leq n \).

Here, we shall use the second method. To find \( T_4(x) \), we must take terms with \( x^k \), \( k \leq 4 \) in the Taylor series above. But this series contains only even-power terms, so we must take the three terms: \( n = 0, 1, 2 \). (Because \( 2n \leq 4 \iff n \leq 2 \iff n = 0, 1, 2 \).) Let us calculate these three terms:
\[
\begin{align*}
( -2)^0 - \frac{(-1)^0 3^0}{0!} x^0 &= (1 - 1)x^0 = 0; \\
( -2)^1 - \frac{(-1)^1 3^2}{2!} x^2 &= (-2 - (1)9/2)x^2 = \frac{5}{2} x^2; \\
( -2)^2 - \frac{(-1)^2 3^4}{4!} x^4 &= (4 - \frac{81}{24})x^4 = \frac{5}{8} x^4.
\end{align*}
\]

Thus,
\[
T_4(x) = \frac{5}{2} x^2 + \frac{5}{8} x^4.
\]
Appendix. Math 126 Syllabus

This is the third and the most difficult course in the 124/5/6 three quarter sequence. It covers analytic geometry, differential geometry, partial derivatives, double integrals, and Taylor polynomials and series, usually in the order as listed below. Prerequisites: Math 124, Math 125. You should be familiar with differential and integral calculus of functions of one variable. This course is often chosen by future science, engineering, chemistry or biology majors. For math majors, we have another sequence: the Honors Calculus Sequence Math 134/5/6. For business and humanities students, we have yet another sequence: Math 111/112. We use Stewart’s book *Multivariable Calculus*, chapters 10, 12 - 15, and *Taylor Notes*. These are the lecture notes on Taylor polynomials and series, which are used instead of the Chapter 11 of Multivariable Calculus. By default, the sections in the syllabus below relate to the Stewart’s book.

1. Analytic Geometry
   - Three-dimensional Cartesian coordinates and vectors (12.1-2)
   - Dot and cross product (12.3-4)
   - Lines and planes (12.5)
   - Quadric surfaces (12.6)

2. Differential Geometry
   - Parametric Curves (10.1, 13.1)
   - Derivatives and integrals of vector functions (10.2, 13.2)
   - Polar coordinates (10.3)
   - Arc length, TNB basis and curvature (13.3)
   - Motion in the space: velocity and acceleration (13.4)

3. Multivariable Differential Calculus
   - Functions of two and more variables (14.1)
   - Partial derivatives of first and higher orders (14.3)
   - Tangent planes and linear approximation (14.4)
   - Local maxima and minima of functions of two variables (14.7)

4. Multivariable Integral Calculus
   - Definition of double integrals (15.1, 15.3)
   - Double and iterated integrals (15.2)
   - Double integrals in polar coordinates (15.4)
   - Applications of double integrals (15.5)

5. Taylor Polynomials and Taylor Series (Taylor Notes)
   - Linear and quadratic approximation (sections 1-2)
   - Taylor approximation in the general case (section 3)
   - Taylor series (sections 4-5)