• Your exam should consist of this cover sheet, followed by 6 problems.

• Unless otherwise indicated, show all your work and justify your answers. Unless otherwise indicated, your answers should be exact values rather than decimal approximations. For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.

• You may use a scientific calculator: TI-30X, and one double-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices, including graphing or programmable calculators, and calculators which can do calculus, are forbidden.

• The use of headphones, earbuds during the exam is not permitted. Turn off all your electronic devices and put them away.

• If you need more space, write on the back and indicate this. If you still need more space, raise your hand and I’ll give you some extra paper to staple onto the back of your test.

• Academic misconduct will guarantee a score of zero on this exam. DO NOT CHEAT.

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1. (10 points) You need to select a governing committee board for an insurance company. This board should contain 4 actuaries, 4 experts in finance, and one CEO. There are 20 candidates: 12 actuaries and 8 experts in finance, each of whom is qualified to be a CEO. How many ways are there to select this board? (You do not need to calculate this up to a decimal number.)

Solution: Suppose you select an actuary to be a CEO. You can do this in 12 ways. Then you are left with 11 actuaries aiming for 4 positions and 8 finance experts aiming for 4 positions. Therefore, the number of ways to do this is

\[ 12 \binom{11}{4} \binom{8}{4}. \]

Alternatively, assume you select a finance expert to be a CEO. There are 8 candidates. You are left with 7 candidates aiming for 4 positions in finance and 12 actuaries aiming for 4 positions. The number of ways to do this is

\[ 8 \binom{7}{4} \binom{12}{4}. \]

The total number of ways to select the board is

\[ 12 \binom{11}{4} \binom{8}{4} + 8 \binom{7}{4} \binom{12}{4}. \]

Another way to solve this problem: you select 4 actuaries out of 12, which you can do in \( \binom{12}{4} \) ways, and 4 finance experts out of 8, in \( \binom{8}{4} \) ways. Then you select a CEO out of the remaining 20 – 4 – 4 = 12 people, which gives you 12 ways to do this. All in all, you have the following answer:

\[ \binom{12}{4} \binom{8}{4} 12 \]
2. (10 points) Find the sum

\[ S = \binom{25}{13} + \binom{25}{14} + \ldots + \binom{25}{24}. \]

**Solution:** Note that

\[ \binom{25}{13} = \binom{25}{12}, \quad \binom{25}{14} = \binom{25}{11}, \quad \binom{25}{15} = \binom{25}{10}, \ldots, \quad \binom{25}{24} = \binom{25}{1}. \]

and so

\[ 2S = \binom{25}{1} + \binom{25}{2} + \ldots + \binom{25}{24}. \]

However,

\[ \binom{25}{0} + \binom{25}{1} + \binom{25}{2} + \ldots + \binom{25}{24} + \binom{25}{25} = 2^{25}. \]

In addition,

\[ \binom{25}{0} = \binom{25}{25} = 1. \]

Therefore,

\[ 2S = 2^{25} - 2, \quad S = 2^{24} - 1. \]
3. (10 points) Consider three events, $A$, $B$, and $C$. Find an example of events $A, B, C$ such that $A$ and $B$ are independent, $B$ and $C$ are independent, $A$ and $C$ are independent, but $A, B$ and $C$ are not independent.

**Solution:** Toss a coin twice. Then $\Omega = \{HH, HT, TH, TT\}$ with $P(HH) = P(HT) = P(TH) = P(TT) = 1/4$. Consider the following events: $A = \{\text{first H}\}$, $B = \{\text{second H}\}$, $C = \{\text{the same both times}\}$. Then

$$P(A) = P(B) = P(C) = \frac{1}{2},$$

and

$$A \cap B = A \cap C = B \cap C = \{HH\}.$$ 

Therefore,

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}.$$ 

This means that each two events are independent. However, $A, B$ and $C$ are not independent: $A \cap B \cap C = \{HH\}$; therefore,

$$P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B) \cdot P(C).$$
4. (10 points) Suppose there is a rare disease which affects 1% of people. There is a test which can
tell whether a person has this disease or not. If the person is healthy, the test shows he is healthy
with probability 90%. If the person has this disease, the test shows this with probability 80%.
A random person is selected from the population, and the test is applied twice. Both times, it
shows that the person has this disease. What is the probability that this person actually has
this disease?

**Solution:** Let $F_1 = \{\text{healthy}\}$, $F_2 = \{\text{sick}\}$, $A = \{\text{twice tested sick}\}$. Then

$$P(F_1) = 0.99, \quad P(F_2) = 0.01,$$

and

$$P(A \mid F_1) = (1 - 0.9)^2 = 0.1^2 = 0.01, \quad P(A \mid F_2) = 0.8^2 = 0.64.$$

Therefore, by Bayes’ formula,

$$P(F_2 \mid A) = \frac{P(A \mid F_2) P(F_2)}{P(A \mid F_2) P(F_2) + P(A \mid F_1) P(F_1)} = \frac{0.64 \cdot 0.01}{0.64 \cdot 0.01 + 0.99 \cdot 0.01} = \frac{64}{163} \approx 39\%$$
5. (10 points) Consider a collateralized debt obligation (CDO) backed by ten subprime mortgages. Five of them are from California, each of which defaults with probability 50%. The other five mortgages are from Florida, each of which defaults with probability 60%. A senior tranch in this CDO defaults only if all of these mortgages default. Find the probability that the senior tranch does not default in the following two cases:

(a) these mortgages all default independently of each other;

(b) all five California mortgages default or do not default simultaneously, and the same is true for the five Florida mortgages, but California defaults are independent of Florida defaults.

Solution: (a) The probability that all of these ten mortgages default is equal to $0.5^5 \cdot 0.6^5 = 0.00243 = 0.243\%$. The probability that at least one of them does not default (and hence the senior tranch does not default) is $99.757\%$.

(b) Now, the probability that at least one of them does not default is $1 - 0.5 \cdot 0.6 = 70\%$. 
6. (10 points) An auto owner can purchase a collision coverage and a disability coverage. These purchases are independent of each other. He is twice as likely to purchase a collision coverage than a disability coverage. The probability that he purchases both is 15%. What is the probability that he purchases neither? (This is a problem from an old actuarial exam in Probability.)

Solution: Let \( H = \{ \text{purchases collision coverage} \} \) and \( D = \{ \text{purchases disability coverage} \} \). Denote \( P(D) = x \). Then \( P(H) = 2x \), and by independence \( P(H \cap D) = 2x \cdot x = 2x^2 \). But this probability is known to be 15% = 0.15, so

\[
2x^2 = 0.15 \implies x = \sqrt{0.075}.
\]

By the inclusion-exclusion formula,

\[
P(H \cup D) = P(H) + P(D) - P(H \cap D) = 2x + x - 2x^2 = 3x - 2x^2 = 3 \sqrt{0.075} - 2 \cdot 0.075 \approx 67.2%.
\]

Thus, the probability that he purchases neither is \( 1 - 67.2\% = 32.8\% \).