• Your exam should consist of this cover sheet, followed by 6 problems.

• Unless otherwise indicated, show all your work and justify your answers. Unless otherwise indicated, your answers should be exact values rather than decimal approximations. For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.

• You may use a scientific calculator, as the TI-30X, and one double-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices, including graphing or programmable calculators, and calculators which can do calculus, are forbidden.

• The use of headphones, earbuds during the exam is not permitted. Turn off all your electronic devices and put them away.

• If you need more space, write on the back and indicate this.

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1. (10 points) A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let $X$ denote the number of luxury cars sold in a given day, and let $Y$ denote the number of extended warranties sold. We have:

$$\begin{align*}
  P(X = 0, \ Y = 0) &= \frac{1}{6}, & P(X = 1, \ Y = 0) &= \frac{1}{12}, & P(X = 2, \ Y = 0) &= \frac{1}{12}, \\
  P(X = 1, \ Y = 1) &= \frac{1}{6}, & P(X = 2, \ Y = 1) &= \frac{1}{3}, & P(X = 2, \ Y = 2) &= \frac{1}{6}.
\end{align*}$$

Calculate the variance of $X$. (From a former Probability actuarial exam.)

**Solution:** Let us find the distribution of $X$:

$$\begin{align*}
  P(X = 0) &= \frac{1}{6}, \\
  P(X = 1) &= \frac{1}{12} + \frac{1}{6} = \frac{1}{4}, \\
  P(X = 2) &= \frac{1}{12} + \frac{1}{3} + \frac{1}{6} = \frac{7}{12}.
\end{align*}$$

Therefore,

$$\begin{align*}
  \mathbb{E}X &= 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{7}{12} = \frac{1}{4} + \frac{7}{6} = \frac{17}{12}, \\
  \mathbb{E}X^2 &= 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{7}{12} = \frac{31}{12}, \\
  \text{Var} \ X &= \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{31}{12} - \left(\frac{17}{12}\right)^2 = \frac{83}{144} = 0.5764.
\end{align*}$$
2. (10 points) A company sells car insurance. There are two types of drivers: \( N_1 = 5000 \) good drivers, and \( N_2 = 3000 \) bad drivers. This year, each good driver gets into an accident with probability \( p_1 = 0.01\% \), and each bad driver gets into an accident with probability \( p_2 = 0.03\% \). All accidents can happen independently of one another. Find approximately the probability that there will be at least two accidents this year.

**Solution:** We can model the random variable \( X \), the number of accidents, using Poisson distribution with parameter \( \lambda = N_1 p_1 + N_2 p_2 = 1.4 \). So we can find

\[
P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda} = 0.4082
\]
3. (10 points) A public health researcher examines the medical records of a group of 937 men who passed away in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease. Calculate the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease. (From a former Probability actuarial exam.)

Solution: Among men who did not have parents suffering from heart disease, $210 - 102 = 108$ died from causes related to heart disease. There are $937 - 312 = 625$ men who had parents neither of whom suffered from heart disease. Therefore, the conditional probability is $\frac{108}{625} = 0.1728$. 
4. (10 points) Suppose you take a sequence a Bernoulli trials with probability \( p = 0.4 \) of success. Let \( X \) be the number of trials you need to make to get 4 successful trials. Find \( E_X \) and \( \text{Var} \, X \).

**Solution:** This random variable \( X \) has negative binomial distribution \( \text{NB}(r, p) \) with \( r = 4 \) and \( p = 0.4 \), and the probability of failure in one Bernoulli trial is \( q = 1 - p = 0.6 \). Therefore,

\[
E_X = \frac{r}{p} = 10 \quad \text{and} \quad \text{Var} \, X = \frac{rq}{p^2} = 15
\]
5. (10 points) Toss six fair coins. Let \( X \) be the number of Heads in the first two tosses. Let \( Y \) be the number of Tails in the last four tosses. Find \( E(X - Y) \) and \( \text{Var}(X - Y) \).

**Solution:** These two random variables are independent, because \( X \) depends on tosses 1 and 2, and \( Y \) depends on tosses 3, 4, 5, 6. Also, \( X \sim \text{Bin}(2, 0.5) \) and \( Y \sim \text{Bin}(4, 0.5) \). We know that for a Binomial random variable \( Z \sim \text{Bin}(N, p) \), we have: \( E(Z) = Np \), and \( \text{Var}(Z) = Np(1 - p) \).

Therefore,
\[
E X = 2 \cdot 0.5 = 1, \quad E Y = 4 \cdot 0.5 = 2, \\
\text{Var} X = 2 \cdot 0.5 \cdot (1 - 0.5) = 0.5, \quad \text{Var} Y = 4 \cdot 0.5 \cdot (1 - 0.5) = 1.
\]

Thus, we have:
\[
E(X - Y) = E X - E Y = 1 - 2 = -1
\]

Also, for independent random variables \( X \) and \( Y \), we have:
\[
\text{Var}(X - Y) = \text{Var} X + \text{Var}(-Y) = \text{Var} X + \text{Var} Y = 0.5 + 1 = 1.5
\]

We used the fact that
\[
\text{Var}(-Y) = E(-Y)^2 - (E(-Y))^2 = EY^2 - (EY)^2 = \text{Var} Y.
\]
6. (10 points) Consider two random variables $X$ and $Y$ with joint distribution

$$P(X = -1, Y = 0) = P(X = 1, Y = 0) = P(X = 0, Y = 1) = P(X = 0, Y = -1) = \frac{1}{4}.$$ 

Are they independent? Find $\text{Cov}(X, Y)$.

**Solution:** These random variables are not independent, because if you know that $Y = 1$, this determines the value of $X$. To put it differently,

$$P(Y = 1, X = 1) = 0, \text{ but } P(Y = 1) \cdot P(X = 1) \neq 0.$$

Next,

$$EX = (-1) \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = 0,$$

and similarly $EY = 0$. For all cases, $XY = 0$ (because always one of the variables $X$ and $Y$ is equal to zero), and therefore $E(XY) = 0$. Thus,

$$\text{Cov}(X, Y) = E(XY) - (EX)(EY) = 0 - \frac{1}{3} \cdot 0 = 0$$

These random variables have zero correlation but they are dependent.