Your exam should consist of this cover sheet, followed by 6 problems.

Unless otherwise indicated, show all your work and justify your answers. Unless otherwise indicated, your answers should be exact values rather than decimal approximations. For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.

You may use a scientific calculator: TI-30X, and one double-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices, including graphing or programmable calculators, and calculators which can do calculus, are forbidden.

The use of headphones, earbuds during the exam is not permitted. Turn off all your electronic devices and put them away.

If you need more space, write on the back and indicate this. If you still need more space, raise your hand and I’ll give you some extra paper to staple onto the back of your test.

Academic misconduct will guarantee a score of zero on this exam. DO NOT CHEAT.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Total:</td>
<td>60</td>
<td></td>
</tr>
</tbody>
</table>
1. (10 points) Consider the Markov chain with transition matrix

\[
A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}
\]

Find the probability that, starting from 3, it hits 2 before hitting 1.

**Solution:** Let \( p_x \) be the probability that, starting from \( x \), the Markov chain hits 2 before 1. Then

\[
p_1 = 0, \quad p_2 = 1, \quad p_3 = 0.1p_1 + 0.2p_2 + 0.7p_3.
\]

Therefore,

\[
p_3 = 0.2 + 0.7p_3 \implies p_3 = \frac{2}{3}
\]
2. (10 points) Consider the Markov chain with transition matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0.5 & 0.5 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

Which states are transient and which are recurrent? Is it irreducible? What is its period? Find all stationary distributions.

**Solution:** State 4 is transient, all other states are recurrent. If you ignore the transient state 4, this Markov chain is irreducible (because states 1, 2, and 3 communicate), and aperiodic (because you can get from 3 to 3 in one step). If

\[
p = [p_1 \ p_2 \ p_3 \ p_4]
\]

is a stationary distribution, then it satisfies the equation \( p = pA \), which is

\[
\begin{align*}
p_1 &= 0.5p_3 + p_4 \\
p_2 &= p_1 \\
p_3 &= p_2 + 0.5p_3 \\
p_4 &= 0
\end{align*} \quad \Rightarrow \quad \begin{align*}
p_1 &= 0.5p_3 \\
p_2 &= p_1 \\
p_4 &= 0
\end{align*}
\]

Combining it with \( p_1 + p_2 + p_3 + p_4 = 1 \), we have:

\[
p_1 = p_2 = \frac{1}{4}, \ p_3 = \frac{1}{2}, \ p_4 = 0.
\]

Therefore, the only stationary distribution is

\[
[\frac{1}{4} \ \frac{1}{4} \ \frac{1}{2} \ 0]
\]
3. (10 points) Consider the Markov chain with transition matrix

\[
A = \begin{bmatrix}
0.5 & 0 & 0.5 & 0 \\
0.3 & 0.2 & 0.5 & 0 \\
0.4 & 0 & 0.6 & 0 \\
0.3 & 0.2 & 0.1 & 0.4 \\
\end{bmatrix}
\]

Find all transient states. For each transient state \( i \), find the distribution of time \( T_i \) spent in \( i \) if the Markov chain starts from \( i \). In addition, find \( E[T_i] \).

**Solution:** Transient states: 2 and 4. The Markov chain can go from 4 to 4 and from 4 to 2 (as well as to the recurrent states 1 and 3), but it can go from 2 only to 2 (and to the recurrent states 1 and 3). Therefore, \( T_2 \) is geometric with parameter 0.8:

\[
P(T_2 = n) = 0.8 \cdot 0.2^{n-1}, \quad n = 1, 2, \ldots, \quad E[T_2] = \frac{1}{0.8} = 1.25 = \frac{5}{4}
\]

Similarly, \( T_4 \) is geometric with parameter 0.6:

\[
P(T_4 = n) = 0.6 \cdot 0.4^{n-1}, \quad n = 1, 2, \ldots, \quad E[T_4] = \frac{1}{0.6} = \frac{5}{3}
\]
4. (10 points) Consider the Markov chain with transition matrix

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.6 & 0.2 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}$$

It has eigenvalues

$$\lambda_1 = -0.6, \ \lambda_2 = 0.4, \ \lambda_3 = 1,$$

and eigenvectors

$$v_1 = [-1 \ 0 \ 1], \ v_2 = [-1 \ 2 \ -1], \ v_3 = [1 \ 1 \ 1]$$

Find its stationary distribution and rate of convergence.

**Solution:** The stationary distribution $p$ is the normalized eigenvector $v_3$ corresponding to the eigenvalue $\lambda_3 = 1$. It should be normalized so that the sum of its components is 1. So we have:

$$p = \frac{1}{1 + 1 + 1} v_3 = \left[ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \right]$$

The rate of convergence is given by the largest eigenvalue (in terms of absolute value) except $\lambda_3 = 1$. This means $0.6^n$.
5. (10 points) Consider an example of two restaurants, McDonalds and Subway. Every day, each person goes to one and only one restaurant. On the 0th day (initially) all people go to McDonalds, because Subway has not yet opened. Every day, 70% of people who went to McDonalds the previous day go to Subway, and 30% return back to McDonalds. Similarly, 50% of people who went to Subway the previous day return there, and 50% go to McDonalds. A meal in McDonalds costs 5$, and a meal in Subway costs 7$. Find the distribution of the money spent by a random person on a second day, and the mean of this money.

Solution: Suppose $X_n$ is the restaurant a random person went to on the $n$th day: either it is $M$ (McDonalds) or $S$ (Subway). Then $X = (X_0, X_1, X_2, \ldots) = (X_n)_{n \geq 0}$ is a Markov chain with transition matrix

$$A = \begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.5 \end{bmatrix}$$

We assume that the first row and column correspond to McDonalds, and the second row and column correspond to Subway. Let

$$x(n) = \begin{bmatrix} P(X_n = M) \\ P(X_n = S) \end{bmatrix}$$

be the distribution of choices of this person on the $n$th day. Then $x(0) = [1 \ 0]$, and

$$x(1) = x(0)A = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix},$$

$$x(2) = x(1)A = \begin{bmatrix} 0.3 \cdot 0.3 + 0.7 \cdot 0.5 & 0.3 \cdot 0.7 + 0.7 \cdot 0.5 \end{bmatrix} = \begin{bmatrix} 0.44 & 0.56 \end{bmatrix}$$

Therefore, on the second day the random person goes to McDonalds with probability 44%, and spends 5$ there. With probability 56%, he goes to Subway, and spends 7$ there. If $Z$ is the amount of money spent, then

$$\mathbb{P}(Z = 5) = 0.44, \ \mathbb{P}(Z = 7) = 0.56$$

In addition,

$$\mathbb{E}Z = 5 \cdot 0.44 + 7 \cdot 0.56 = 6.12$$
6. (10 points) An actuary discovered that policyholders are three times as likely to file two claims as to file four claims. The number $X$ of claims filed has a Poisson distribution. Calculate $\text{Var } X$. (This is a problem from an old actuarial exam in Probability.)

**Solution:** If $X \sim \text{Poi}(\lambda)$ with an unknown $\lambda$, then

$$P(X = 2) = \frac{\lambda^2}{2!} e^{-\lambda}, \quad P(X = 4) = \frac{\lambda^4}{4!} e^{-\lambda}.$$ 

Therefore,

$$3 = \frac{P(X = 2)}{P(X = 4)} = \frac{4!}{2! \lambda^2} = \frac{12}{\lambda^2}.$$

We get: $\lambda = 2$, and $\text{Var } X = \lambda = 2$.