Suppose \(\xi_1, \xi_2, \xi_3, \ldots\) is a sequence of independent identically distributed random variables, \(\xi_k = \pm 1\) with equal probability \(1/2\). Then \(S_n := \sum_{k=1}^n \xi_k\) is called a random walk. It starts from 0 and increases or decreases per \(1\) with probability \(1/2\) each time.

Let us introduce a "continuous - time infinitesimal random walk". First of all, define \(S_t\) for non-integer \(t\) by linear interpolation between \(S_k\) and \(S_{k+1}\), where \(k < t < k + 1\). Then set \(S_t^{(m)} = \frac{1}{\sqrt{m}} S_{mt}\). Why do we necessarily need to multiply by \(1/\sqrt{m}\)? Because

\[
\text{Var} S_n = \text{Var} \sum_{k=1}^n \xi_k = \sum_{k=1}^n \text{Var} \xi_k = \sum_{k=1}^n 1 = n.
\]

Therefore, \(\text{Var} S_t \approx \text{Var} S_k = k \approx t\), where \(k < t < k + 1\). And \(\text{Var} S_{mt} = mt\). But \(\text{Var}(\alpha S_{mt}) = \alpha^2 \text{Var} S_{mt}\). We need to ensure \(\text{Var}(\alpha S_{mt}) = t\); so it is necessary to take \(\alpha = 1/\sqrt{m}\).

So \(S_t^{(m)}\) is a random walk with one move (of length \(1/\sqrt{m}\)) at each \(1/m\) unit of time. This is a continuous function \(\mathbb{R}_+ \to \mathbb{R}\). Now let us do the most important part. Let \(m \to \infty\).

**Donsker - Prohorov’s principle.** As \(m \to \infty\), the distribution of \(S_t^{(m)}\) converges (on each interval \([0,T]\)), to the distribution of some (random) function \(\mathbb{R}_+ \to \mathbb{R}\), which is called a Brownian motion or Wiener process and is denoted by \(B_t, B(t), W_t, W(t)\).

**Basic properties of Brownian motion.**

1. \(W(0) = 0\), because \(S_0^{(m)} = 0\).
2. **Independent increments.** Suppose \(0 < t_1 < t_2 < \ldots < t_n\). Then

\[
W(t_n) - W(t_{n-1}), \ldots, W(t_2) - W(t_1), W(t_1)
\]

are independent random variables.

Indeed, for any \(k = 1, \ldots, n\) we have:

\[
W(t_k) - W(t_{k-1}) \approx \frac{1}{\sqrt{m}} \sum_{j=mt_{k-1}+1}^{tm_k} \xi_j,
\]

and all \(\xi_j\) are independent.

3. \(W(t) - W(s) \sim \mathcal{N}(0, t-s)\), where \(\mathcal{N}(0, \sigma^2)\) is a **Gaussian distribution** with mean zero, variance \(\sigma^2\) and density

\[
\frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}.
\]

Indeed, recall the Central Limit Theorem (or the de-Mouvre - Laplace theorem) which says that

\[
\frac{1}{\sqrt{m}} \sum_{k=1}^m \xi_k \to N(0, 1),
\]

and observe:

\[
W(t) - W(s) = \lim_{m \to \infty} \frac{1}{\sqrt{m}} (S_{mt} - S_{ms}) = \lim_{m \to \infty} \frac{\sqrt{t-s}}{\sqrt{m(t-s)}} \sum_{j=ms+1}^{mt} \xi_j = \sqrt{t-s} \mathcal{N}(0, 1) = \mathcal{N}(0, t-s).
\]
4. $W$ is a continuous function, since it is a uniform limit (in distributional sense) of continuous
functions.

These four properties are usually set as a definition (i.e. as axioms) in Probability courses, and
then the existence theorem is proved, and another properties are deduced from these axioms.

**Other properties of Brownian motion.**
1. $W$ is not differentiable anywhere. By the way, it is rather hard to explicitly construct a
continuous nowhere differentiable function.
2. $W$ has infinite variation on any interval $[0, T]$; that is,
$$\sup \sum_{k=1}^{n} |W(t_k) - W(t_{k-1})| = \infty,$$
where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_n = T$ of $[0, T]$.
Recall that if $f \in C^1[0, T]$, then this variation is $\int_0^T |f'(t)| dt$.
But $W$ has finite quadratic variation: that is,
$$\sum_{k=1}^{n} (W(t_k) - W(t_{k-1}))^2 \to T$$
as the diameter $\max(t_k - t_{k-1})$ of a partition tends to 0.
3. **Law of iterated logarithm.**
$$\lim_{t \to \infty} \frac{W(t)}{\sqrt{2 \ln \ln t}} = 1.$$  
Since $W$ and $-W$ have the same law, we have:
$$\lim_{t \to \infty} \frac{W(t)}{\sqrt{2 \ln \ln t}} = -1.$$ 
So this strange function oscillates between $\pm \sqrt{2 \ln \ln t}$.

**Applications.**
1. Financial modeling. How to model stock prices? Bachelier (1900) suggested a Brownian
motion. In fact, this was the birth of this random process. But Samuelson in 1965 suggested a
much better (although not perfect) model: *geometric Brownian motion* $e^{\mu t + \sigma W(t)}$. In 1973 Black,
Scholes and Merton did research in option pricing using this model and got a Nobel Prize in
Economics in 1997 for this research.
2. Random movement of small particles (which explains its name!)
3. Numerous other applications in modern physics.

**References.**
2. Ekkenard Kopp. *From Measures to Ito Integrals.*
4. Ito and McKean. *Diffusion Processes and Their Sample Paths.*