Optional Stopping Theorem
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Suppose every minute you toss a symmetric coin. If it comes heads (with probability 1/2), you win 1$. If it comes tails (also with probability 1/2), you lose 1$. Let $X_k$ be your win (or loss) at the moment $k$. So $X_k$ takes values $\pm 1$ with equal probability. All $X_k$ are independent. Your total capital at the moment $n$ is

$$S_n := \sum_{k=1}^{n} X_k, \quad S_0 = 0.$$

This random process $S = (S_n)_{n \geq 0}$ is called a random walk. You can stop this game at any moment. This moment needs not be deterministic; it can be random. Denote this moment by $\tau$. This is a nonnegative integer-valued random variable. And your total gain in this game will be $S_\tau$. This $\tau$ is your strategy: you cannot influence the values of $X_k$, but you can choose when to withdraw yourself from this game. It is also called a stopping time.

You can decide whether to exit the game at the moment $n$ only basing on the past: using the values of $X_1, \ldots, X_n$, which you already know by this moment. Speaking formally, the event $\{\tau = n\}$ depends only on $X_1, \ldots, X_n$.

So we consider the random walk $S = (S_n)_{0 \leq n \leq N}$. There are two cases: when the time horizon $N$ is finite and when it is infinite.

Can we find a strategy $\tau$ such that we will gain something? But we will gain some random sum, so let us use the expectation. Question:

**can we find a strategy $\tau$ such that $E X_\tau > 0$?**

**Theorem 1. [Optional Stopping Theorem]** *For finite time horizon, this is not possible: for every strategy $\tau$, we have $E S_\tau = 0$.***

**Lemma.** For every $n = 0, \ldots, N$ we have:

$$E \left( S_n I_{\{\tau = n\}} \right) = E \left( S_N I_{\{\tau = n\}} \right). \quad (1)$$

Suppose we proved this lemma. Let us complete the proof. The variable $\tau$ attains values $0, \ldots, N$. Therefore,

$$E S_\tau = \sum_{n=0}^{N} E S_n I_{\{\tau = n\}} = \sum_{n=0}^{N} E \left( S_N I_{\{\tau = n\}} \right).$$

Recall that for any event $A$, the indicator of $A$ is the random variable which is 1 when $A$ happens and 0 when $A$ does not happen. It follows from (1) that

$$\sum_{n=0}^{N} E \left( S_n I_{\{\tau = n\}} \right) = \sum_{n=0}^{N} E \left( S_N I_{\{\tau = n\}} \right) = E \left( S_N \sum_{n=0}^{N} I_{\{\tau = n\}} \right) = ES_N = 0.$$ 

The proof is complete. Now we only need to prove the lemma.

**Proof of the lemma.** It suffices to prove that $E \left( (S_N - S_n) I_{\{\tau = n\}} \right) = 0$. Note that

$$S_N - S_n = \sum_{k=n+1}^{N} X_k,$$
so it depends only on \(X_{n+1}, \ldots, X_N\). And \(\{\tau = n\}\) depends only on \(X_1, \ldots, X_n\). So \(S_N - S_n\) and \(I_{\{\tau = n\}}\) are independent. Recall: for independent variables \(X, Y\), we have \(\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)\). And

\[
\mathbb{E}(S_N - S_n) = \mathbb{E}\left(\sum_{k=n+1}^{N} X_k\right) = \sum_{k=n+1}^{N} \mathbb{E}X_k = 0.
\]

Thus,

\[
\mathbb{E}\left((S_N - S_n)I_{\{\tau = n\}}\right) = \mathbb{E}\left((S_N - S_n)\right)\mathbb{E}I_{\{\tau = n\}} = 0.
\]

This completes the proof of the lemma and of the theorem.

In the case of infinite time horizon, the situation is different. The random walk will eventually come to the value 1. You should wait until it comes there and then exit the game. Your gain will be 1. Formally speaking, \(\tau = \min\{n \mid S_n = 1\}\), and \(S_\tau = 1\). Certainly, the main question is whether \(\tau\) always exists. Does this random walk eventually come to 1?

**Theorem 2.** With probability 1, there exists \(n\) such that \(S_n = 1\).

**Proof.** Let us split the proof into five steps:
Step 1. Consider the event \(A = \{S_n \geq \sqrt{n} \text{ for infinitely many } n\}\). Then \(\mathbb{P}(A) > 0\).
Step 2. Consider the event

\[
B = \left\{\lim_{n \to \infty} \frac{S_n}{\sqrt{n}} \geq 1\right\}.
\]

Then \(\mathbb{P}(B) > 0\).
Step 3. Fix \(N\). Then \(B\) does not depend on \(X_1, \ldots, X_N\).
Step 4. \(\mathbb{P}(B) = 0\) or 1.
Step 5. Finish the proof.

**Proof of step 1.** Let \(A_m = \{S_m \geq \sqrt{m}\}\). Then

\[
A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \bigcap_{n=1}^{\infty} B_n, \quad B_n = \bigcup_{m=n}^{\infty} A_m.
\]

The sequence of events \(B_1, B_2, \ldots\) is decreasing: \(B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots\). Therefore, \(\mathbb{P}(A) = \lim \mathbb{P}(B_n)\) (this is a property of probability). Note that, by Central Limit Theorem,

\[
\mathbb{P}(B_n) \geq \mathbb{P}(A_n) = \mathbb{P}\{S_n \geq \sqrt{n}\} = \mathbb{P}\left\{\frac{S_n}{\sqrt{n}} \geq 1\right\} \to \mathbb{P}\{\xi \geq 1\},
\]

as \(n \to \infty\). In the first inequality, we used the fact that \(B_n \supseteq A_n\). It suffices to note that

\[
\mathbb{P}\{\xi \geq 1\} = \frac{1}{\sqrt{2\pi}} \int_{1}^{\infty} e^{-x^2/2} dx \approx 0.16 > 0.
\]

Thus, \(\mathbb{P}(A) = \lim \mathbb{P}(B_n) \geq \mathbb{P}\{\xi \geq 1\} > 0\).

**Proof of step 2.** Since \(A \subseteq B\), we have: \(0 < \mathbb{P}(A) \leq \mathbb{P}(B)\).

**Proof of step 3.**

\[
\lim_{n \to \infty} \frac{S_n}{\sqrt{n}} = \lim_{n \to \infty} \left(\frac{1}{\sqrt{n}} \sum_{k=1}^{N} X_k + \frac{S_n - S_N}{\sqrt{n}}\right) = \lim_{n \to \infty} \frac{S_n - S_N}{\sqrt{n}},
\]
and

\[ S_n - S_N = \sum_{k=N+1}^{n} X_k \]

is independent of \( X_1, \ldots, X_N \). Therefore, \( \lim_{n \to \infty} \frac{S_n}{\sqrt{n}} \) does not depend on \( X_1, \ldots, X_N \), and the event \( B \) is also independent of these variables. Such events are called \textit{tail events}.

\textbf{Proof of step 4.} It follows from step 3 that \( B \) is independent of any event generated by \( X_1, \ldots, X_N \), for any \( N \). Let \( N \to \infty \). Then \( B \) is independent of any event generated by \( X_1, X_2, \ldots \). But \( B \) is itself generated by these variables; so \( B \) is independent of itself! Recall that any two events \( C_1, C_2 \) are independent if \( P(C_1 \cap C_2) = P(C_1)P(C_2) \). Let \( C_1 = C_2 = B \). Then \( P(B) = P^2(B) \), and \( P(B) = 0 \) or \( 1 \). This is called \textit{Kolmogorov’s 0–1 law}.

\textbf{Proof of step 5.} Combining the results of steps 2 and 4, we get: \( P(B) = 1 \). Thus, with probability 1, we have:

\[ \lim_{n \to \infty} \frac{S_n}{\sqrt{n}} \geq 1. \]

Therefore, with probability 1, \( S_n > 0 \) for some \( n \). But this random walk starts from 0. Thus, with probability 1, there exists \( n \) such that \( S_n = 1 \).

\textbf{Remark.} You can actually see that \( S \) attains \textit{every integer value infinitely many times}. This can be easily seen from the fact that (by symmetry of the random walk)

\[ \lim_{n \to \infty} \frac{S_n}{\sqrt{n}} \geq 1, \quad \lim_{n \to \infty} \frac{S_n}{\sqrt{n}} \leq -1. \]