Default Contagion in a Banking System

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January 25, 2017
Description of the System

\( \mathcal{N}(t) \): number of banks at time \( t \geq 0 \)
\( X_1(t), \ldots, X_{\mathcal{N}(t)}(t) \): capitals of banks at time \( t \)
\( S(t) := X_1(t) + \ldots + X_{\mathcal{N}(t)}(t) \): total size of the system

Banks can emerge and default (disappear) at random times

**Financial Contagion**: If a bank defaults, capitals of other banks decrease by a random fraction

Between emergence (births) and defaults, each bank behaves as a geometric Brownian motion with drift \( r \) and diffusion \( \sigma^2 \), independently of others
Take a system of $N$ banks with total size $s$

**Births:** $\lambda_N(s)$: intensity of birth
$\mathcal{B}_{N,s}$: distribution of the size of the newly created bank

**Defaults:** $\kappa_N(x_i, s)$: default intensity of the $i$th bank with capital $x_i$
$\mathcal{D}_{N,x_i,s}$: **contagion measure:** for $j \neq i$, $X_j$ decreases by share $\xi_{ij} \sim \mathcal{D}_{N,x_i,s}$, independent for different $j$

If $t$ is the default time, then $X_j(t) = X_j(t-)(1 - \xi_{ij})$
Our system is a Markov process:

$$X = (X(t), \ t \geq 0), \ X(t) = (X_1(t), \ldots, X_N(t))$$

State space \( \mathcal{X} = \bigcup_{k=0}^{\infty} (0, \infty)^k \)

\((0, \infty)^0 = \{\emptyset\} \), corresponding to an empty system (no banks)

An element \( x = (x_1, \ldots, x_n) \in \mathcal{X} \) has dimension \( n(x) := n \), and size \( s(x) := x_1 + \ldots + x_n \)

\( \mathcal{L} \): generator of this process
Figure: Sample paths of $X, S, N$ with $N(0) := 5$, $\sigma := 0.2$, $r := 0.05$. Default rate is $\kappa_n(x, s) = 0.1/(0.01 + x)$, the contagion $D_{n,x,s}$ is the uniform distribution on $[0, 0.2]$, the birth rate is constant $\lambda_n(s) = 1$, and the distribution $B_{n,s}$ of the capital of a new bank is $Exp(1)$, for every $n$ and $s$. The red marker $X$ in the left figure represents default.
Questions of Interest

Non-explosion: Existence of the system over infinite time horizon

Stability of the system: Existence and uniqueness of the stationary distribution; Long-term convergence to the stationary distribution

Large-scale limit: Behavior of the empirical measure for a large number of banks

Individual banks: Behavior of (one) selected bank or a few selected banks in a large system
When does this system not explode? In other words, when is it defined for all \( t \geq 0 \), up to an infinite time horizon?

**Lyapunov function:** \( V : \mathcal{X} \to [0, \infty) \) such that

\[
\mathcal{L} V(x) \leq c_1 V(x) + c_2 \text{ and } V(\infty) = \infty
\]

If it exists, then no explosion

Often can take \( V(x) = n(x) + s(x) \)

This gives us a general condition on the parameters

**Example:** \( \lambda_N(s) = C_1 N \) and \( \kappa_N(x_i, s) = C_2 \)
Stable system: a unique stationary distribution $\Pi$ on $\mathcal{X}$, and

$$\sup_A \left| \mathbb{P}_x(X(t) \in A) - \Pi(A) \right| \to 0 \quad \forall \ x \in \mathcal{X} \ \text{as} \ t \to \infty$$

Assume we find a Lyapunov function $V : \mathcal{X} \to [0, \infty)$ with $V(\infty) = \infty$ and $\mathcal{L}V(x) \leq -C$ for $x \in \mathcal{X}$ outside of a compact set. Then the system is stable.

We can again try $V(x) = n(x) + s(x)$.

This gives us a general condition on the parameters.

Example: $\lambda_N(s) = C_1 N$ and $\kappa_N(x_i, s) = C_2$ with $C_1 < C_2$, with $B_{n,s}$ and $D_{n,x,s}$ depending only on $n$. 
Empirical measure: \( \mu_t := \frac{1}{\mathcal{N}(t)} \sum_{i=1}^{\mathcal{N}(t)} \delta_{X_i(t)} \)

Sequence of systems \( X^{(N)} \) starting from \( N \) banks at time \( t = 0 \), with empirical measures \( \mu_t^{(N)} \)

Our goal: To find the limit of measure-valued processes

\[
\mu^{(N)} = (\mu_t^{(N)})_{0 \leq t \leq T} \quad \text{in} \quad D([0, T], \mathcal{W}_p)
\]

\( \mathcal{W}_p, \ p > 1 \): Wasserstein distance on \( \mathbb{R} \)
Wasserstein distance of order $p$ between probability measures $\nu'$ and $\nu''$ on $\mathbb{R}$:

$$\mathcal{W}_p(\nu', \nu'') = \left[ \inf \mathbb{E} |\xi' - \xi''|^p \right]^{1/p},$$

where the inf is taken over all couplings $(\xi', \xi'')$ of $(\nu', \nu'')$.

For a metric space $E$, $D([0, T], E)$ is the Skorohod space of functions $[0, T] \to E$ which are right-continuous with left limits $D[0, T] := D([0, T], \mathbb{R})$.
Fix a $p > 1$. Then as $N \to \infty$, we need:

- $N^{-1} \lambda_N(Ns) \to \lambda_\infty(s)$
- $\mathcal{B}_{N, Ns} \to \mathcal{B}_\infty, s$ in $\mathcal{W}_p$
- $\kappa_N(x, Ns) \to \kappa_\infty(x, s)$
- For $\xi_{N,x,s} \sim \mathcal{D}_{N,x,s}$, $N\xi_{N,x,Ns} \to \xi_\infty, x, s \sim \mathcal{D}_\infty, x, s$ in $\mathcal{W}_p$

All convergence is uniform in $x, s$.

The limits are continuous and bounded in $x, s$, and there are other additional technical conditions.

Means of limiting measures $\mathcal{B}_\infty, s$ and $\mathcal{D}_\infty, x, s$: $\overline{\mathcal{B}}_\infty(s)$ and $\overline{\mathcal{D}}_\infty(x, s)$

$\psi(x, s) := r - \overline{\mathcal{D}}_\infty(x, s) \kappa_\infty(x, s)$
Under the above conditions, the measure-valued processes $\mu^{(N)}$ converge weakly in $D([0, T], \mathcal{W}_q)$ for $q \in [1, p)$ to a deterministic measure-valued function $\mu^{(\infty)} = (\mu_t^{(\infty)})_{0 \leq t \leq T}$, which is a solution to the McKean-Vlasov jump-diffusion on $\mathbb{R}$ with drift and diffusion

$$\psi(x, m(\nu))x, \quad \frac{1}{2}\sigma^2 x^2,$$

and jump measure:

$$\lambda_{\infty}(m(\nu))B_{\infty,m(\nu)} + (\nu, \kappa_{\infty}(\cdot, m(\nu)))\nu$$

$m(\nu) = \text{mean of probability measure } \nu$
A classic jump-diffusion $Z = (Z(t), t \geq 0)$, behaves as a solution of a stochastic differential equation with drift $g(\cdot)$, diffusion $\sigma^2(\cdot)$:

$$dZ(t) = g(Z(t)) \, dt + \sigma(Z(t)) \, dW(t)$$

between jumps, which occur at time $t$ with intensity $\lambda(Z(t))$ and destinations governed by the measure $Q_{Z(t)}(\cdot)$

Finite measure $\nu_x(\cdot) = \lambda(x)Q_x(\cdot)$: jump measure. Generator:

$$Af(x) = g(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) + \int [f(y) - f(x)] \, \nu_x(dy)$$

McKean-Vlasov jump-diffusions: $g$, $\sigma$, $\nu$ depend not only on $x = Z(t)$ (current position of the process), but on $\mu(t)$, the distribution of $Z(t)$
Births and Defaults in Large-Scale Limit

Jump measure:

$$\lambda_\infty (m(\nu)) B_\infty, m(\nu) + (\nu, \kappa_\infty (\cdot, m(\nu))) \nu$$

$m(\nu) =$ mean of probability measure $\nu$

Births translate into the first component of the jump measure, corresponding to $B_\infty, m(\nu)$

Defaults translate into the second component of the jump measure, corresponding to $\kappa_\infty$

Contagion effect translates into an additional drift in the drift term:

$$\psi(x, y) = r - \overline{D}_\infty(x, s) \kappa_\infty(x, s)$$

Reduces capital of every remaining bank by an i.i.d. share
Assume the above conditions

The mean of $X^{(N)}(t)$ converges in $D[0, T]$ to the mean of the limiting measure

\[
\frac{1}{\mathcal{N}_N(t)} \sum_{i=1}^{\mathcal{N}_N(t)} X_i^{(N)}(t) \to m(t) := m\left(\mu^{(\infty)}\right)
\]
Illustration of the Large-Scale Limit

Figure: Left panel: distribution of the mean of $\mu_t^{(N)}$ for $0 \leq t \leq T$ and $N = 5, 25, 100$, confidence interval [5%, 95%], initial condition $\text{Exp}(0.5)$. Right panel: distribution of $d_N := (\mu_T^{(N)}, 1_{[0,1]})$, $T = 10$, proportion of banks with capital less than 1.
Apply a test function $f$ to the empirical measure:

$$
\left( \mu_t^{(N)}, f \right) = \frac{1}{\mathcal{N}_N(t)} \sum_{i=1}^{\mathcal{N}_N(t)} f \left( X_{i}^{(N)}(t) \right)
$$

Apply Itô’s formula and decompose into the $\int_0^t \ldots \, ds$ part and the local martingale part

Prove tightness of the family of measure-valued processes

The local martingale part converges to zero

Thus, the limit is a deterministic measure-valued function
When $\mathcal{D}_{\infty, x, s}$ and $\kappa_{\infty}(x, s)$ do not depend on $x$, we can solve this McKean-Vlasov jump-diffusion explicitly

$$dZ(t) = \psi(m(t))Z(t)\,dt + \sigma Z(t)\,dB(t)$$

with $m(t) = \mathbb{E}[Z(t)]$, jumps with rate $\lambda_{\infty}(m(t))$ to $y \sim B_{\infty, m(t)}$

First, solve the ordinary differential equation for the mean $m(t)$:

$$m'(t) = \psi(m(t))m(t) + \lambda_{\infty}(m(t)) (\overline{B}_{\infty}(m(t)) - m(t))$$

Then plug this into the above equation for $Z$
When $D_{\infty,x,s}$ and $\kappa_{\infty}(x,s)$ do not depend on $x$, we can find time-independent regime (stationary distribution) for this McKean-Vlasov jump-diffusion explicitly.

First, solve for the (constant) mean $M$:

$$\psi(M)M + \lambda_{\infty}(M) (\overline{B}_{\infty}(M) - M) = 0$$

The distribution of the geometric Brownian motion at a fixed time is lognormal. The resulting geometric Brownian motion, killed at a constant rate, gives us a mixture of lognormal distributions.
The simplest case is when $\lambda_\infty$, $\kappa_\infty$, $\overline{B}_\infty$, $\overline{D}_\infty$ are constants

$$m'(t) = \lambda_\infty \overline{B}_\infty - (r - \overline{D}_\infty \kappa_\infty - \lambda_\infty) m(t)$$

Denote $\gamma := r - \lambda_\infty - \overline{D}_\infty \kappa_\infty$ and assume $\gamma \neq 0$. Then

$$m(t) = \left( m(0) - \frac{\lambda_\infty \overline{B}_\infty}{\gamma} \right) \exp[-\gamma t] + \frac{\lambda_\infty \overline{B}_\infty}{\gamma}$$

Stationary solution is given by

$$M = \frac{\lambda_\infty \overline{B}_\infty}{\gamma}$$
Under conditions of the main theorem: Large-Scale Limit, assume the initial conditions converge: $X_1^{(N)}(0) \to x_1$ as $N \to \infty$

As $N \to \infty$, we have: $X_1^{(N)} \to Y$ in $D[0, T]$ with $Y$ driven by the following stochastic differential equation:

$$dY(t) = \psi(Y(t), m(t))Y(t)\,dt + \sigma Y(t)\,dB(t),$$

killed with rate $\kappa_\infty(Y(t), m(t))$, starting from $Y(0) = x_1$

$m(t)$ is the mean of $\mu_t^{(\infty)}$, already known from McKean-Vlasov jump-diffusion
Fix a $k \geq 1$, the number of banks

As $N \to \infty$, the first $k$ banks $X_1^{(N)}, \ldots, X_k^{(N)}$ converge to independent copies of the aforementioned killed diffusion

Dependence vanishes in the limit: Propagation of Chaos