Collisions of Competing Brownian Particles

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Background
We have $N$ Brownian particles on the real line:

$$X_1(t), \ldots, X_N(t).$$

Rank them from bottom to top:

$$X_{(1)}(t) \leq X_{(2)}(t) \leq \ldots \leq X_{(N)}(t).$$

The particle which is currently the $k$th leftmost has rank $k$.

**Main Rule:** the particle which currently has rank $k$ moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$.

When particles exchange ranks, they also exchange drift and diffusion coefficients.
When two or more particles collide, how to assign ranks?

We resolve ties in favor of the lexicographic order: assign lower ranks to particles $X_i$ with lower indices $i$.

So the SDE governing dynamics of the particles is

$$dX_i(t) = \sum_{k=1}^{N} 1(X_i \text{ has rank } k \text{ at time } t) (g_k dt + \sigma_k dW_i(t)).$$
Stochastic Portfolio Theory (Banner, Fernholz, Karatzas, 2005).

In real world, stocks with smaller capitalizations have larger growth rates and larger volatilities.

Just let

\[ g_1 > g_2 > \ldots > g_N \quad \text{and} \quad \sigma_1 > \sigma_2 > \ldots > \sigma_N, \]

and let

\[ e^{X_1(t)}, \ldots, e^{X_N(t)} \]

be the capitalizations of stocks at time \( t \).
We denote
\[ Y_k(t) \equiv X_{(k)}(t) \]
the \textit{kth ranked particle}. By definition,
\[ Y_1(t) \leq Y_2(t) \leq \ldots \leq Y_N(t). \]
A triple collision:

\[ Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t). \]

A simultaneous collision:

\[ Y_k(t) = Y_{k+1}(t), \quad \text{and} \quad Y_l(t) = Y_{l+1}(t), \quad k \neq l. \]

A triple collision is a particular case of a simultaneous collision.
Motivation: Strong Solutions

\[ dX_i(t) = \sum_{k=1}^{N} 1(X_i \text{ has rank } k \text{ at time } t) (g_k dt + \sigma_k dW_i(t)) . \]

This equation has weak solution which is unique in distribution (Bass, Pardoux, 1987).

It is known to have strong solution, which is pathwise unique, only up to the first moment of a triple collision. (Fernholz, Ichiba, Karatzas, Prokaj, 2013; Ichiba, Karatzas, Shkolnikov, 2013)
Main Results
Condition to Avoid Triple Collisions

Theorem (Ichiba, Karatzas, Shkolnikov, 2011; S, 2014)

There are a.s. no triple collisions if and only if the sequence

\[ \sigma_1^2,\ \sigma_2^2,\ \ldots,\ \sigma_N^2 \]

is concave, that is,

\[ \sigma_k^2 \geq \frac{1}{2} (\sigma_{k-1}^2 + \sigma_{k+1}^2), \quad k = 2, \ldots, N - 1. \]

If this condition is violated for some \( k \), then with positive probability there is a triple collision between

\[ Y_{k-1},\ Y_k,\ Y_{k+1}. \]
Lack of Simultaneous Collisions

Theorem (S, 2014)

If the sequence

$$\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2$$

is concave, then there are also a.s. no simultaneous collisions.

An interesting corollary: if there are a.s. no triple collisions, then there are a.s. no simultaneous collisions.
For the rest of this section, consider a system of $N = 4$ particles,

$$Y_1(t) \leq Y_2(t) \leq Y_3(t) \leq Y_4(t).$$

What do we need to avoid total collisions?

$$Y_1(t) = Y_2(t) = Y_3(t) = Y_4(t).$$
Total Collisions for $N = 4$ Particles

**Theorem (S, 2014)**

If the following conditions hold:

\[
\begin{align*}
9\sigma_1^2 &\leq 7\sigma_2^2 + 7\sigma_3^2 + 7\sigma_4^2; \\
3\sigma_1^2 &\leq 5\sigma_2^2 + \sigma_3^2 + \sigma_4^2; \\
3\sigma_1^2 + 3\sigma_4^2 &\leq 5\sigma_2^2 + 5\sigma_3^2; \\
3\sigma_4^2 &\leq \sigma_1^2 + \sigma_2^2 + 5\sigma_3^2; \\
9\sigma_4^2 &\leq 7\sigma_1^2 + 7\sigma_2^2 + 7\sigma_3^2,
\end{align*}
\]

then a.s. there are no total collisions.
Suppose we are interested in avoiding triple collisions of the type

\[ Y_1(t) = Y_2(t) = Y_3(t). \]

**Theorem (S, 2014)**

*If the five inequalities above together with*

\[ \sigma_2^2 \geq \frac{1}{2} (\sigma_1^2 + \sigma_3^2) \]

*hold, then a.s. there are no collisions of the type*

\[ Y_1(t) = Y_2(t) = Y_3(t). \]
Comparison of Results

Let

\[ \sigma_1^2 = \sigma_2^2 = \sigma_4^2 = 1, \quad \sigma_3^2 = 0.9. \]

Then there are a.s. no triple collisions of the type

\[ Y_1(t) = Y_2(t) = Y_3(t), \]

but the concavity condition is violated:

\[ \sigma_3^2 < \frac{1}{2} (\sigma_2^2 + \sigma_4^2), \]

so with positive probability there is a triple collision of the type

\[ Y_2(t) = Y_3(t) = Y_4(t). \]
Now, let us discuss avoiding simultaneous collisions of the type

\[ Y_1(t) = Y_2(t), \quad Y_3(t) = Y_4(t). \]

**Theorem (S, 2014)**

*If the five inequalities above hold, then a.s. there are no collisions of the type*

\[ Y_1(t) = Y_2(t), \quad Y_3(t) = Y_4(t). \]
Let

\[ \sigma_1^2 = \sigma_4^2 = 1, \quad \sigma_2^2 = \sigma_3^2 = 0.9. \]

Then there are a.s. no simultaneous collisions of the type

\[ Y_1(t) = Y_2(t), \quad Y_3(t) = Y_4(t), \]

but the concavity condition is violated:

\[ \sigma_2^2 < \frac{1}{2} (\sigma_1^2 + \sigma_3^2), \quad \sigma_3^2 < \frac{1}{2} (\sigma_2^2 + \sigma_4^2), \]

so there is a triple collision of each type with positive probability:

\[ Y_1(t) = Y_2(t) = Y_3(t) \quad \text{and} \quad Y_2(t) = Y_3(t) = Y_4(t). \]
Sketch of Proof
This is the $\mathbb{R}^{N-1}_+$-valued process

$$Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))',$$

defined as

$$Z_k(t) = Y_{k+1}(t) - Y_k(t).$$
A triple collision

\[ Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t) \]

is equivalent to

\[ Z_{k-1}(t) = Z_k(t) = 0. \]

A simultaneous collision

\[ Y_k(t) = Y_{k+1}(t), \quad Y_l(t) = Y_{l+1}(t) \]

is equivalent to

\[ Z_k(t) = Z_l(t) = 0. \]

So there are no triple and simultaneous collisions if and only if the gap process does not hit non-smooth parts of the boundary.
It turns out that the gap process $Z$ is a version of a semimartingale reflected Brownian motion (SRBM) in the orthant $\mathbb{R}^d_+$:

An SRBM in the orthant with reflection matrix $R$, drift vector $\mu$ and covariance matrix $A$ is a Markov process in $\mathbb{R}^d_+$ which:

- moves as a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$ inside the orthant;
- on each face $\{x \in \mathbb{R}^d_+ | x_i = 0\}$, it is reflected according to the vector $r_i$, the $i$th column of $R$.

If $r_i = e_i$, then the reflection is normal. Otherwise, it is oblique.
Gap Process is an SRBM

The gap process is an SRBM in $\mathbb{R}^{N-1}_+$ with reflection matrix

$$R = \begin{bmatrix}
1 & -1/2 & 0 & \ldots \\
-1/2 & 1 & -1/2 & \ldots \\
0 & -1/2 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots \ldots
\end{bmatrix}$$

covariance matrix

$$A = \begin{bmatrix}
\sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \ldots \\
-\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & \ldots \\
0 & -\sigma_3^2 & \sigma_3^2 + \sigma_4^2 & \ldots \\
\ldots & \ldots & \ldots & \ldots \ldots
\end{bmatrix}$$

and drift vector

$$\mu = (g_2 - g_1, g_3 - g_2, \ldots, g_N - g_{N-1})'.$
Consider an $\text{SRBM}^d(R, \mu, A)$. Under some additional technical conditions, we have the following theorem:

**Theorem (S, 2013)**

For $R = (r_{ij})_{1 \leq i, j \leq d}$ and $A = (a_{ij})_{1 \leq i, j \leq d}$, the SRBM does not hit non-smooth parts of the boundary if and only if

$$r_{ij}a_{jj} + r_{ji}a_{ii} \geq 2a_{ij}, \quad 1 \leq i < j \leq d.$$  

If this condition is violated for some $i < j$, then with positive probability the SRBM hits the edge

$$\{x_i = x_j = 0\}.$$  

Based on results by Williams (1987).
Take $Z = \text{SRBM}^d(R, \mu, A)$.

If $R$ is symmetric and invertible, then (under some additional technical assumptions) for

$$F : \mathbb{R}_+^d \to \mathbb{R}_+, \quad F(z) := \left[ z' R^{-1} z \right]^{1/2}$$

we have: $Z$ hits the corner of the orthant if and only if $F(Z(t))$ hits zero.

It turns out that in the SDE for $F(Z(t))$, there are no boundary terms, this is an Itô process. We can compare it with Bessel process and find whether it hits zero or not.
For $I \subseteq \{1, \ldots, d\}$, define the edge of the boundary

$$S_I := \{ x \in \mathbb{R}^d_+ | x_i = 0, \ i \in I \}.$$ 

**Theorem (Ichiba, Karatzas, Shkolnikov, 2013; S, 2014)**

An SRBM$^d(R, \mu, A)$ does not hit $S_I$ a.s. if for every $J$ such that

$$I \subseteq J \subseteq \{1, \ldots, d\}$$

we have: an SRBM$^{\left|J\right|}(\left[R\right]_J, \left[\mu\right]_J, \left[A\right]_J)$ does not hit the corner a.s.

Here, $\left[R\right]_J = (r_{ij})_{i,j \in J}$, $\left[A\right]_J = (a_{ij})_{i,j \in J}$, $\left[\mu\right]_J = (\mu_j)_{j \in J}$.

This allows us to consider individual collisions such as

$$Y_1(t) = Y_2(t) \text{ and } Y_3(t) = Y_4(t).$$
We can translate these results from the language of an SRBM to the language of competing Brownian particles. This requires quite a bit of calculations with matrices $R$ and $A$. 
Asymmetric Collisions
Go back to the original system. If $L_{(k,k+1)}$ is the local time process of $Y_{k+1} - Y_k$ at zero: local time of collision between $Y_k$ and $Y_{k+1}$, then

$$Y_k(t) = Y_k(0) + g_k t + \sigma_k B_k(t) - \frac{1}{2} L_{(k,k+1)}(t) + \frac{1}{2} L_{(k-1,k)}(t).$$

For convenience, $L_{(0,1)} \equiv 0$ and $L_{(N,N+1)} \equiv 0$.

The meaning of coefficients $1/2$ is that the local time of collision is split evenly between colliding particles.
Change coefficients $1/2$ to some other coefficients, get a system with asymmetric collisions:

\[ Y_k(t) = Y_k(0) + g_k t + \sigma_k B_k(t) - q_k^- L_{(k,k+1)}(t) + q_k^+ L_{(k-1,k)}(t). \]

(Karatzas, Pal, Shkolnikov, 2015; Fernholz, Ichiba, Karatzas, 2013)

Must satisfy for each $k$

\[ 0 < q_k^\pm < 1, \quad q_{k+1}^+ + q_k^- = 1. \]

Methods from this talk also work for such systems.
This is a system $Y$ of ranked particles.

A system $X = (X_1, \ldots, X_N)'$ of named particles:

$$\text{d}X_i(t) = 1(X_i \text{ has rank } k \text{ at time } t) \left( g_k \text{d}t + \sigma_k \text{d}W_i(t) \right)$$

$$- \left( q^-_k - (1/2) \right) \text{d}L(k,k+1)(t) + \left( q^+_k - (1/2) \right) \text{d}L(k-1,k)(t).$$

is known to exist (in a strong sense) only up to the first moment of a triple collision.

**Open Problem:** Does it exist after a triple collision?
Thank You!