Concentration of Measure for Stochastic Heat Equation

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Joint work with Davar Khoshnevisan
Order \( p \geq 1 \).

Metric space \((\mathcal{X}, d)\).

Two probability measures \( \mathbb{P} \) and \( \mathbb{Q} \) on \( \mathcal{X} \).

Wasserstein Distance of Order \( p \):

\[
W_p(\mathbb{P}, \mathbb{Q}) = \inf \left[ \mathbb{E} d^p(X, Y) \right]^{1/p}
\]

where the infimum is taken over all couplings \((X, Y) \sim (\mathbb{P}, \mathbb{Q})\).

Convergence in \( W_p \) = weak convergence + convergence of \( p \)th moments.
Relative Entropy

Metric space $(\mathcal{X}, d)$.

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $\mathcal{X}$.

Relative Entropy:

$$\mathcal{H}(\mathbb{Q} \mid \mathbb{P}) = -\mathbb{E}_{\mathbb{P}} \frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}$$

if $\mathbb{Q} \ll \mathbb{P}$ and $\infty$ otherwise.

This is a generalization of entropy of the distribution $(p_1, \ldots, p_n)$:

$$H(p) = -p_1 \log p_1 - \ldots - p_n \log p_n.$$
We write $\mathbb{P} \in T_p(C)$ if for every $Q \ll \mathbb{P}$ we have:

$$\mathcal{W}_p(\mathbb{P}, Q) \leq \sqrt{2C \mathcal{H}(Q \mid \mathbb{P})}.$$ 

We say $\mathbb{P}$ satisfies transportation-cost information inequality or Talagrand concentration inequality of order $p$ with constant $C$.

For $1 \leq q < p$, $T_p(C) \subseteq T_q(C)$. 
Gaussian Tail Estimate: If $\mathbb{P} \in T_1(C)$, then for any 1-Lipschitz function $f : \mathcal{X} \to \mathbb{R}$ with median $m(f)$ we have:

$$\mathbb{P}(|f - m(f)| \geq \delta) \leq 2 \exp(-\delta^2/8C), \quad \delta \geq 2\sqrt{2C \log 2}.$$

Marton (1996)

Tensorization: If $\mathbb{P}, \mathbb{Q} \in T_2(C)$, then $\mathbb{P} \times \mathbb{Q} \in T_2(C)$ with appropriate norm. This property holds only for order 2.

Ledoux (2001)
The following list is far from complete:

**General theory:**
- Talagrand (1996)
- Bobkov, Gotze (1999)
- Ledoux (2001)

**Stochastic differential equations:**
- Pal (2012)
- Cattiaux, Guillin (2013)
- Pal, Shkolnikov (2014)

**Applications:**
- Massart (2007): model selection
- Dubhashi, Panconesi (2012): randomized algorithms
- Lacker (2015): stochastic finance
Unknown function: \( u(t, x), t \geq 0, 0 \leq x \leq 1. \)

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + g(x, u) + \dot{W}(t, x).
\]

Initial condition: \( u|_{t=0} = u_0(x), \) deterministic.

Boundary condition: \( u|_{x=0} = u|_{x=1} = 0. \)

Space-time white noise: \( W(t, x), \) “flickers” of independent noise at every point \((t, x)\).

Drift coefficient \( g: \) \( |g(x, u) - g(x, v)| \leq L|u - v|. \)

Solution exists and is unique, is a.s. continuous.
The distribution of $u$ satisfies $T_2(C)$ in the space of continuous functions $u : [0, T] \times [0, 1] \to \mathbb{R}$ with the supremum norm, and with

$$C = 2\pi^{-1/2} \sqrt{T} \exp(2L^2 T^2).$$

(Khoshnevisan, S, 2017)
Similar results hold for:
More general operators instead of $\frac{\partial^2}{\partial x^2}$: fractional derivative, second-order differential operators.
Additional diffusion term $\sigma(x, u) \dot{W}$.
Different norms (integral instead of maximum).
Neumann boundary conditions.
Many dimensions.
Space-colored instead of space-white noise.