Collisions of Competing Brownian Particles

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We have $N$ Brownian particles on the real line:

$$X_1(t), \ldots, X_N(t).$$

Rank them from left to right:

$$X_{(1)}(t) \leq X_{(2)}(t) \leq \ldots \leq X_{(N)}(t).$$

The particle which is currently the $k$th leftmost has rank $k$.

**Main Rule:** the particle which currently has rank $k$ moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$.

When particles exchange ranks, they also exchange drift and diffusion coefficients.
$Y_k(t) \equiv X_{(k)}(t)$: the $k$th ranked particle. By definition,

$$Y_1(t) \leq Y_2(t) \leq \ldots \leq Y_N(t).$$

A triple collision:

$$Y_{k-1}(t) = Y_k(t) = Y_{k+1}(t).$$

A simultaneous collision:

$$Y_k(t) = Y_{k+1}(t), \quad \text{and} \quad Y_l(t) = Y_{l+1}(t), \quad k \neq l.$$

A triple collision is a particular case of a simultaneous collision. If these were independent Brownian motions, then there would be no such collisions. But here, matters are different.
A formal way to write an SDE for competing Brownian particles is

$$dX_i(t) = \sum_{k=1}^{N} 1(X_i \text{ has rank } k \text{ at time } t) (g_k dt + \sigma_k dW_i(t)).$$

This equation has a weak solution which is unique in distribution (Bass, Pardoux, 1987).

It is known to have a strong solution, which is pathwise unique, only up to the first moment of a triple collision (Ichiba, Karatzas, Shkolnikov, 2013).
Theorem (Ichiba, Karatzas, Shkolnikov, 2011; S, 2014)

There are a.s. no triple and no simultaneous collisions if and only if the sequence

\[ \sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2 \]

is concave, that is,

\[ \sigma_k^2 \geq \frac{1}{2} (\sigma_{k-1}^2 + \sigma_{k+1}^2) , \quad k = 2, \ldots, N - 1. \]

If this condition is violated for some \( k \), then with positive probability there is a triple collision between

\[ Y_{k-1}, Y_k, Y_{k+1}. \]
Simultaneous Collisions for $N = 4$ Particles

**Theorem (S, 2014)**

\[
\begin{align*}
9\sigma_1^2 & \leq 7\sigma_2^2 + 7\sigma_3^2 + 7\sigma_4^2; \\
3\sigma_1^2 & \leq 5\sigma_2^2 + \sigma_3^2 + \sigma_4^2; \\
3\sigma_1^2 + 3\sigma_4^2 & \leq 5\sigma_2^2 + 5\sigma_3^2; \\
3\sigma_4^2 & \leq \sigma_1^2 + \sigma_2^2 + 5\sigma_3^2; \\
9\sigma_4^2 & \leq 7\sigma_1^2 + 7\sigma_2^2 + 7\sigma_3^2;
\end{align*}
\]

Then a.s. there are no collisions of the type

\[Y_1(t) = Y_2(t) \text{ and } Y_3(t) = Y_4(t).\]

**Theorem (Bruggeman, 2014)**

\[\sigma_1^2 + \sigma_4^2 \leq \sigma_2^2 + \sigma_3^2\] is also sufficient for the above.
Suppose we are interested in avoiding triple collisions of the type

\[ Y_1(t) = Y_2(t) = Y_3(t). \]

**Theorem (S, 2014)**

*If the five inequalities above together with*

\[ \sigma_2^2 \geq \frac{1}{2} (\sigma_1^2 + \sigma_3^2) \]

*hold, then a.s. there are no collisions of the type*

\[ Y_1(t) = Y_2(t) = Y_3(t). \]
Let

\[ \sigma_1^2 = \sigma_2^2 = \sigma_4^2 = 1, \quad \sigma_3^2 = 0.9. \]

Then there are a.s. no triple collisions of the type

\[ Y_1(t) = Y_2(t) = Y_3(t), \]

and no simultaneous collisions of the type

\[ Y_1(t) = Y_2(t) \quad \text{and} \quad Y_3(t) = Y_4(t), \]

but the concavity condition is violated:

\[ \sigma_3^2 < \frac{1}{2} (\sigma_2^2 + \sigma_4^2), \]

so with positive probability there is a triple collision of the type

\[ Y_2(t) = Y_3(t) = Y_4(t). \]
Thank You!