Infinite Systems of Competing Brownian Particles

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We have $N$ Brownian particles on the real line:

$$X_1(t), \ldots, X_N(t).$$

Rank them from bottom to top:

$$X_{(1)}(t) \leq X_{(2)}(t) \leq \ldots \leq X_{(N)}(t).$$

The particle which is currently the $k$th lowest has rank $k$.

**Main Rule:** the particle which currently has rank $k$ moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$.

When particles exchange ranks, they also exchange drift and diffusion coefficients.
When two or more particles are tied, how to assign ranks?

We resolve ties in favor of the lexicographic order: assign lower ranks to particles $X_i$ with smaller indices $i$.

So the SDE governing dynamics of the particles is

$$dX_i(t) = \sum_{k=1}^{N} 1(X_i \text{ has rank } k \text{ at time } t)(g_k dt + \sigma_k dW_i(t)).$$

This system was introduced in (Banner, Fernholz, Karatzas, 2005). It has a weak solution, unique in law (Bass, Pardoux, 1987).
Motivation

- Stochastic Portfolio Theory
  (Banner, Fernholz, Karatzas, 2005; Pal, Chatterjee, 2010)
- Similar systems with totally asymmetric collisions
  (Ferrari, Spohn, Weiss, 2015)
- Scaling limits of asymmetrically colliding random walks
  (Karatzas, Pal, Shkolnikov, 2012)
- Discretized version of McKean-Vlasov equation
  (Jourdain, Reygner, 2013)
  (Dembo, Shkonlikov, Varadhan, Zeitouni, 2012)
A **triple collision:**

\[ X_{(k-1)}(t) = X_{(k)}(t) = X_{(k+1)}(t). \]

A **simultaneous collision:**

\[ X_{(k)}(t) = X_{(k+1)}(t), \text{ and } X_{(l)}(t) = X_{(l+1)}(t), \quad k \neq l. \]

A triple collision is a particular case of a simultaneous collision, because we can let \( l = k + 1. \)

This SDE is known to have a **strong solution**, which is pathwise unique, **only up to the first moment of a triple collision** (Ichiba, Karatzas, Shkolnikov, 2013). Whether there is a strong solution after this moment is an open question.
Theorem (Ichiba, Karatzas, Shkolnikov, 2013; S, 2014)

There are a.s. no triple and simultaneous collisions if and only if the sequence \((\sigma_k^2)\) is concave:

\[ \sigma_k^2 \geq \frac{1}{2} (\sigma_{k-1}^2 + \sigma_{k+1}^2), \quad k = 2, \ldots, N - 1. \]

If this condition does not hold for a certain \(k\), then with positive probability there exists a triple collision between particles with ranks \(k - 1, k, k + 1\).

**Corollary:** If there are a.s. no triple collisions, then there are a.s. no simultaneous collisions.
This is the process taking values in the orthant $\mathbb{R}^{N-1}_+:$

$$Z(t) = (Z_1(t), \ldots, Z_{N-1}(t))',$$

defined as follows: for $k = 1, \ldots, N - 1, \ t \geq 0,$

$$Z_k(t) = X_{(k+1)}(t) - X_{(k)}(t).$$

This is a semimartingale reflected Brownian motion in the orthant.

- In the interior of the orthant, it behaves as an $(N - 1)$-dimensional Brownian motion with certain drift vector and covariance matrix.
- At each face $\{z_k = 0\}$ of the boundary, it is reflected back in the orthant, not normally, but obliquely.
There exists a stationary distribution $\pi$ for the gap process if and only if

$$\bar{g}_k > \bar{g}_N, \quad k = 1, \ldots, N - 1,$$

that is, if the average drift for a few bottom particles is greater than the average drift for all particles.

If this stationary distribution exists, then it is unique.

Regardless of the initial condition,

$$Z(t) \Rightarrow \pi, \quad \text{as} \quad t \to \infty.$$

(Pal, Pitman, 2008; Ichiba et al 2011)
Stationary Distribution for the Gap Process

In case of the skew-symmetry condition

\[
\sigma_k^2 - \sigma_{k-1}^2 = \sigma_{k+1}^2 - \sigma_k^2, \quad k = 2, \ldots, N - 1,
\]

the stationary distribution \( \pi \) is the product of exponentials:

\[
\pi = \left( \prod_{k=1}^{N-1} \left( \frac{4k(\overline{g}_k - \overline{g}_N)}{\sigma_k^2 + \sigma_{k+1}^2} \right) \right).
\]

\( Exp(\alpha) \) is the exponential distribution with rate \( \alpha \) and mean \( \alpha^{-1} \).

(Pal, Pitman, 2008; Ichiba et al 2011)
Infinitely many Brownian particles on the real line: $X_1(t), X_2(t), \ldots$

Rank them from the bottom upward:

$$X_{(1)}(t) \leq X_{(2)}(t) \leq X_{(3)}(t) \leq \ldots$$

Main Rule: the particle which currently has rank $k$ moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$. When particles exchange ranks, they also exchange drift and diffusion coefficients.

$$dX_i(t) = \sum_{k=1}^{\infty} 1(X_i \text{ has rank } k \text{ at time } t) \left( g_k \, dt + \sigma_k \, dW_i(t) \right).$$

Introduced in (Shkolnikov, 2011) and studied further in (Ichiba, Karatzas, Shkolnikov, 2013).
Unlike the finite system, the infinite system might not always exist, even in the weak sense.

**Why?** Because it might not be possible to rank these particles.

Assume for simplicity that \( g_1 = g_2 = \ldots = 0 \) and \( \sigma_1^2 = \sigma_2^2 = \ldots = 1 \). Start infinitely many i.i.d. Brownian motions from the same initial point. Then it is impossible to rank them from bottom to top.

This shows that we need to have some condition on initial values \( x_i = X_i(0), \ i = 1, 2, \ldots \). They should be far enough apart.
Weak Existence and Uniqueness

Theorem (Ichiba, Karatzas, Shkolnikov, 2013; S, 2014)

Suppose for some $n_0$ we have:

$$g_{n_0} = g_{n_0+1} = \ldots \quad \text{and} \quad \sigma_{n_0} = \sigma_{n_0+1} = \ldots$$

Also, $X_i(0) = x_i \to \infty$ fast enough:

$$\sum_{n=1}^{\infty} \exp(-\alpha x_n^2) < \infty \quad \text{for all} \quad \alpha > 0.$$ 

Then the system exists in the weak sense and is unique in law.

E.g. $x_i = ci$, $x_i = c \log i$ work, but $x_i = c(\log i)^{1/2}$ does not.
Theorem (Pal-Pitman, 2008; S, 2015)

Suppose that

\[ \sigma_n = \sigma > 0, \quad n = 1, 2, \ldots \quad \text{and} \quad \sum_{n=1}^{\infty} g_n^2 < \infty, \]

and \( X_i(0) = x_i \to \infty \) fast enough (same condition).

Then the system exists in the weak sense and is unique in law.

Idea of proof: Girsanov’s theorem.
Weak Existence

Theorem (S, 2014)

Suppose that

\[ \sup_{n \geq 1} |g_n| < \infty, \quad \sup_{n \geq 1} \sigma_n^2 < \infty, \]

and \( X_i(0) = x_i \to \infty \) fast enough:

\[ \sum_{n=1}^{\infty} \exp(-\alpha x_n^2) < \infty \quad \text{for all} \quad \alpha > 0. \]

Then the system exists in the weak sense.

However, we do not have uniqueness.

This system is a weak limit point of the corresponding finite systems.
For every $N \geq 2$, consider a system

$$X^{(N)} = \left( X_1^{(N)}, \ldots, X_N^{(N)} \right)'$$

of $N$ competing Brownian particles, starting from $(x_1, \ldots, x_N)$, with drift coefficients $g_1, \ldots, g_N$, diffusion coefficients $\sigma_1^2, \ldots, \sigma_N^2$.

Then there exists a subsequence $(N_j)_{j \geq 1}$ such that for $k = 1, 2, \ldots$ we have weak convergence in $C([0, T], \mathbb{R}^{2k})$, as $j \to \infty$:

$$\left( X_1^{(N_j)}, \ldots, X_k^{(N_j)}, X_1^{(N_j)}, \ldots, X_k^{(N_j)} \right)' \Rightarrow \left( X_1, \ldots, X_k, X_1, \ldots, X_k \right)' .$$

This is how we construct a weak copy of the infinite system.
Theorem (S, 2014)

Under the conditions of the latter theorem, there are no triple and simultaneous collisions if and only if the sequence \((\sigma_1^2, \sigma_2^2, \ldots)\) is concave:

\[
\sigma_k^2 \geq \frac{1}{2} \left( \sigma_{k-1}^2 + \sigma_{k+1}^2 \right), \quad k = 1, 2, \ldots
\]

If this condition is violated for some \(k\), then with positive probability there is a triple collision between particles with ranks \(k - 1, k, k + 1\).

A corollary: if there are a.s. no triple collisions, then there are a.s. no simultaneous collisions.
Suppose for some $n_0$ we have:

$$g_{n_0} = g_{n_0+1} = \ldots \quad \text{and} \quad \sigma_{n_0} = \sigma_{n_0+1} = \ldots$$

Also, $X_i(0) = x_i \to \infty$ fast enough (same condition).

The system exists in the strong sense and is pathwise unique until the first moment of a triple collision. Whether it is true after this moment is unknown (Ichiba, Karatzas, Shkolnikov, 2013).

**Corollary.** If the sequence $(\sigma_k^2)_{k \geq 1}$ is concave, then the system exists in the strong sense and is pathwise unique for $0 \leq t < \infty$. 
The gap process for infinite systems of competing Brownian particles

\[ X_1(t), X_2(t), \ldots \]

is defined similarly: it is an \( \mathbb{R}_+^\infty \)-valued process

\[ Z(t) = (Z_1(t), Z_2(t), Z_3(t), \ldots), \quad t \geq 0, \]

\[ Z_k(t) = X_{(k+1)}(t) - X_{(k)}(t), \]

where \( X_{(k)} \) is the \( k \)th ranked particle.
From now on, consider the case

\[ g_1 = \ldots = g_M = 1, \quad g_{M+1} = g_{M+2} = \ldots = 0, \]

\[ \sigma_1 = \sigma_2 = \ldots = 1. \]

The results are available in a more general setting, but here they are simple. A particular case is infinite Atlas model, with \( M = 1 \):

\[ g_1 = 1, \quad g_2 = g_3 = \ldots = 0; \quad \sigma_1 = \sigma_2 = \ldots = 1. \]

It is called so because the bottom-ranked particle supports all other particles on its shoulders, like the ancient Atlas.
The average drift for a few bottom-ranked particles (which is positive) is greater than the average drift for all particles (which is zero). So we can expect that there is a stationary distribution.

Moreover, the skew-symmetry condition:

$$\sigma^2_{k+1} - \sigma^2_k = \sigma^2_k - \sigma^2_{k-1}, \quad k = 2, 3, \ldots$$

holds for \( \sigma^2_1 = \sigma^2_2 = \ldots = 1 \). So we can expect that this stationary distribution has product-of-exponentials form.
The gap process $Z = (Z(t), t \geq 0)$ has a stationary distribution

$$\pi = \text{Exp}(2) \otimes \text{Exp}(4) \otimes \ldots \otimes \text{Exp}(2M) \otimes \text{Exp}(2M) \otimes \ldots$$

In particular, for the infinite Atlas model we recreate the result by Pal, Pitman (2008):

$$\pi = \bigotimes_{k=1}^{\infty} \text{Exp}(2).$$
Idea of proof: Approximate this infinite system by finite systems of $N$ particles:

$$g_1 = \ldots = g_M = 1, \quad g_{M+1} = g_{M+2} = \ldots = g_N = 0,$$

$$\sigma_1 = \sigma_2 = \ldots = \sigma_N = 1.$$ 

We can find a stationary distribution $\pi^{(N)}$ for the gap process of this system. As $N \to \infty$, each marginal of $\pi^{(N)}$ tends to the marginal of $\pi$. 
Unlike the finite case, we do not know whether this stationary distribution $\pi$ is unique.

We also do not know whether $Z(t) \Rightarrow \pi$ as $t \to \infty$: whether this stationary distribution serves as a limiting distribution. (See partial results later.)

Long-run behavior of the bottom-ranked particle:

$$X_{(1)}(t), \quad t \to \infty.$$ 

For $M = 1$ (infinite Atlas model), it behaves as a fractional Brownian motion with $H = 1/4$ (Dembo, Tsai, 2015). For $M \geq 2$?
For the infinite Atlas model ($M = 1$), there is a family of stationary distributions:

$$\pi_a := \prod_{k=1}^{\infty} \text{Exp}(2 + ka), \quad a \geq 0,$$

which includes the (already known) Pal-Pitman distribution

$$\pi \equiv \pi_0 := \prod_{k=1}^{\infty} \text{Exp}(2).$$

Similar hypothesis can be formulated for other $M \geq 2$. 
Heuristics: The gap process is an ”SRBM in the infinite-dimensional orthant”.

Following (Williams, 1995), we formally calculate the product-of-exponentials distribution, borrowing the technique from finite-dimensional case, where it is proved rigorously.

It turns out that this involves ”inversion of an infinite-dimensional reflection matrix”, which can have many values. From here, we get many candidates $\pi_a, a \geq 0$, for the stationary distribution.

But this is not rigorous.
Two probability measures $P$ and $Q$ on $\mathbb{R}_+^\infty$. We say:

\[ P \preceq Q : \quad Q \text{ stochastically dominates } P, \]

if for all $x_1, x_2, \ldots \geq 0$,

\[ P\left([x_1, \infty) \times [x_2, \infty) \times \ldots \right) \leq Q\left([x_1, \infty) \times [x_2, \infty) \times \ldots \right). \]
Convergence Results

This is a partial result on weak convergence of $Z(t)$ as $t \to \infty$ to the stationary distribution $\pi$. Recall: for finite systems, always $Z(t) \Rightarrow \pi$. For infinite systems, we do not know in general case.

Theorem (S, 2014)

(i) Every weak limit point of $Z(t)$ as $t \to \infty$ (that is, every distribution $\nu$ such that $Z(t_j) \Rightarrow \nu$ for some $t_j \to \infty$) is stochastically smaller than the stationary distribution $\pi$.

(ii) If $\pi \preceq Z(0)$, then $Z(t) \Rightarrow \pi$ as $t \to \infty$.

Part (ii) says that if we start from the gaps wider than $\pi$, then the gap process weakly converges to $\pi$. 
Thanks!