Market Models with Splits and Mergers

Andrey Sarantsev
Joint work with Ioannis Karatzas
University of California, Santa Barbara

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We have $N$ Brownian particles on the real line:

$$X_1(t), \ldots, X_N(t).$$

Rank them from top to bottom:

$$X_{(1)}(t) \geq X_{(2)}(t) \geq \ldots \geq X_{(N)}(t).$$

The particle which is currently the $k$th highest has rank $k$.

**Main Rule:** the particle which currently has rank $k$ moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$.

**CBP-Based Market Model:** Capitalizations of $N$ stocks

$$Y_1(t) = e^{X_1(t)}, \ldots, Y_N(t) = e^{X_N(t)}.$$
Market weight of the $i$th company

\[ \mu_i(t) = \frac{Y_i(t)}{Y_1(t) + \ldots + Y_N(t)} \]

A market model is diverse if $\mu_i(t) \leq 1 - \delta$ for all $i$ and $t$, where $\delta \in (0, 1)$ is fixed.

Such models allow an arbitrage: a portfolio which beats the market (Fernholz, 2002; Fernholz, Karatzas, Kardaras, 2005; Fernholz, Karatzas, 2009).

Fact: The CBP-based market model is not diverse.
Try to amend the CBP model to make it diverse. Allow for a changing number of stocks (Strong, 2011).

**Splits:** When a company’s market weight reaches $1 - \delta$, forcefully split it into two companies of random size.

**Mergers:** Set an exponential clock with rate $\lambda_N$, where $N$ is the current number of companies. When it rings, choose randomly two companies and merge them. If the resulting company has market weight greater than or equal to $1 - \delta$, this merger is suppressed. Otherwise, it is allowed to proceed.
$N$ stocks at time $t$: $(\pi_0(t), \ldots, \pi_N(t))$

- $\pi_0(t) + \ldots + \pi_N(t) = 1$
- $|\pi_i(t)| \leq K$ for $K$ independent of $i, N, t$
- $\pi_0(t)$ is the share of the current capital invested in money
- $\pi_i(t)$ is the share invested in the $i$th stock

When two companies merge, then the portfolio weight corresponding to the new company’s stock is the sum of the portfolio weights corresponding to the two old stocks.

When a company gets split into two new ones, then its weight in the portfolio is partitioned in proportion to the weights of the newly created companies.
An investor starts with initial capital 1 and invests in the stock market according to a portfolio \( \pi \). The **wealth process** \( V^\pi = (V^\pi(t), t \geq 0) \) does not change at the moment of jumps.

A portfolio \( \pi \) represents an **arbitrage opportunity** relative to another portfolio \( \rho \) if

\[
V^\pi(T) \geq V^\rho(T) \quad \text{a.s.}
\]

\[
V^\pi(T) > V^\rho(T) \quad \text{with positive probability}
\]
Main Results

Under certain conditions on parameters,

- This model is non-explosive: the quantity of stocks does not go to infinity in finite time
- An equivalent martingale measure exists, so there is no arbitrage: we cannot beat the market

For market models with constant number of stocks, diversity leads to arbitrage. For varying number of stocks, this is no longer true. (On this, see also Fouque & Strong, 2011)
We choose two companies to be merged as follows. Exclude the company with the largest capitalization. From the remaining $N-1$ companies, choose a subset of two companies uniformly. Each subset is chosen with probability $1/(\binom{N-1}{2})$.

For some parameters $c, \alpha > 0$, the rate of merging:

$$\lambda_N \sim cN^\alpha \quad \text{as} \quad N \to \infty.$$ 

**Additional Condition:** The top stock (with rank 1) has minimal drift coefficient.
Thank You!