Approximation of Reflected Diffusions by Solutions of SDE

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Reflected Brownian Motion on the Half-Line
\[ Z = (Z(t), t \geq 0) \text{ on } \mathbb{R}_+ := [0, \infty). \]

- When \( Z(t) > 0 \), \( Z \) behaves as a Brownian motion.
- When \( Z(t) = 0 \), \( Z \) is reflected back.
- \( Z(t) \) cannot be negative.
We can write $Z(t)$ as

$$Z(t) = W(t) + L(t),$$

where $W$ is a Brownian motion, and $L$ is a nondecreasing random process with $L(0) = 0$ which can increase only when $Z(t) = 0$.

When $Z(t) = 0$ and $Z(t)$ wants to go below zero, $L(t)$ increases so that $Z(t)$ stays nonnegative.

$Z(t)$ cannot penetrate the barrier at zero.
We can think about $L(t)$ as a singular drift at $x = 0$, which creates a hard barrier at zero.

**Question:** Can we approximate this hard barrier by a soft barrier created by the drift:

$$dX(t) = b(X(t))dt + dW(t)?$$

In particular, $b(x) = a1_{[-c,0]}(x)$ for a large $a > 0$, a small $c > 0$. 
Take two sequences \((a_n)\) and \((c_n)\) of positive numbers such that
\[
a_n \to \infty, \quad c_n \to 0, \quad a_n c_n \to \infty.
\]
Take a sequence of stochastic processes
\[
dX_n(t) = a_n 1_{[-c_n,0]}(X_n(t)) dt + dW(t).
\]
Then \(X_n \Rightarrow Z\) in law on \(C[0, T]\).
That is, for every Borel subset \(F \subseteq C[0, T]\) with \(P(Z \in \partial F) = 0\),
\[
P(X_n \in F) \to P(Z \in F).
\]
Main Approximation Result

\[ dX_n(t) = f_n(X_n(t))dt + dW(t). \]

- for all \( \varepsilon > 0 \),
  \[ \int_{-\varepsilon}^{\varepsilon} f_n(z)dz \to \infty. \]
- for all \([x_1, x_2] \subseteq (0, \infty)\),
  \[ \int_{x_1}^{x_2} f_n(z)dz \to 0. \]
- there exists a \( \delta > 0 \) such that \( f_n(x) \geq 0 \) for \( n \) large enough and \( |x| < \delta \).

Then \( X_n \Rightarrow Z \) in law on \( C[0, T] \).
For a real-valued diffusion $X$, a function $s : \mathbb{R} \to \mathbb{R}$ is called a scale function if $s(X)$ is a local martingale.

The scale function $s_{X_n}$ must satisfy the equation

$$\frac{1}{2}s_{X_n}''(x) + f_n(x)s_{X_n}'(x) = 0,$$

because the term with $\,\mathrm{d}t$ in Itô’s formula for $s_{X_n}(X_n(t))$ must be zero. After a correct choice of a particular solution $s_{X_n}$, we prove

$$s_{X_n}(x) \to s_Z(x) := \begin{cases} 
    x, & x > 0; \\
    -\infty, & x < 0, 
\end{cases} \quad n \to \infty.$$
Reflected Diffusions on the Half-Line
A continuous $\mathbb{R}_+\text{-valued}$ process $Z = (Z(t), t \geq 0)$:

- When $Z(t) > 0$, it behaves as a solution of SDE with drift coefficient $g$ and diffusion coefficient $\sigma^2$.
- When $Z(t) = 0$, it is reflected back to the positive half-line.

Formally speaking:

$$Z(t) = Z(0) + \int_0^t g(Z(s))\,ds + \int_0^t \sigma(Z(s))\,dW(s) + L(t),$$

where $L$ is a nondecreasing random process with $L(0) = 0$ which can increase only when $Z(t) = 0$. 
We can rewrite the last slide as

\[ dZ(t) = g(Z(t))dt + \sigma(Z(t))dW(t) + dL(t). \]

Take the following SDE:

\[ dX(t) = f(X(t))dt + \tilde{\sigma}(X(t))dW(t). \]

- \( f(x) \approx g(x) \) for \( x \geq x_0 > 0 \).
- \( f(x) \geq 0 \) is large when \( x \approx 0 \).
- \( \tilde{\sigma}(x) \approx \sigma(x) \) for all \( x \geq 0 \).

Then we can expect \( X \approx Z \).
Consider a sequence of solutions of SDE

\[ dX_n(t) = f_n(X_n(t))dt + \sigma_n(X_n(t))dW(t). \]

- For all \( \varepsilon > 0 \), \( \int_{-\varepsilon}^{\varepsilon} f_n(z)dz \to \infty \).
- For all \([x_1, x_2] \subseteq (0, \infty)\), \( \int_{x_1}^{x_2} |f_n(z) - g(z)|dz \to 0 \).
- There exists a \( \delta > 0 \) s.t. \( f_n(x) \geq g(x) \) for \( n \geq n_0 \) and \( |x| < \delta \).
- For every \( x \geq 0 \), \( \sigma_n \to \sigma \) uniformly on \([0, x]\).
- The family \((\sigma_n)\) is equicontinuous at \( x = 0 \).

Then \( X_n \Rightarrow Z \) in law on \( C[0, T] \).
Multidimensional Reflected Brownian Motion in a Domain
A domain (open connected subset) $D \subseteq \mathbb{R}^d$. Boundary $\partial D$ is smooth, except non-smooth parts $\mathcal{V} \subseteq \partial D$ of the boundary.

**Example:** $D = \mathbb{R}^d_+$ is the orthant, $\partial D$ consists of $d$ faces

$$S_i := \{ x \in S \mid x_i = 0 \}.$$  

Then $\mathcal{V} = \bigcup_{i < j} (S_i \cap S_j)$.

Subset $\mathcal{V}$ is **small**: $\text{dist}(x, \partial D) = \text{dist}(x, \partial D \setminus \mathcal{V})$ for all $x \in \mathbb{R}^d$. 
At every point $x \in \partial D \setminus \mathcal{V}$, inward unit normal vector $n(x)$.

Continuous reflection field $r : \partial D \setminus \mathcal{V} \to \mathbb{R}^d$, $r(x) \cdot n(x) \equiv 1$.

If $r(x) = n(x)$: normal reflection. Otherwise: oblique reflection.
Informal Description

Fix $\mu \in \mathbb{R}^d$ and a positive definite symmetric $d \times d$-matrix $A$.

$Z = (Z(t), t \geq 0)$, a continuous process with values in $\overline{D}$.

- When $Z(t) \in D$, it behaves as a $d$-dimensional Brownian motion with drift vector $\mu$ and covariance matrix $A$.
- When $Z(t) = x \in \partial D \setminus \mathcal{V}$, it is reflected inside $D$ in the direction of the reflection vector $r(x)$.
- When $Z(t) \in \mathcal{V}$, it is stopped.
\[ Z(t) = W(t \wedge \tau_V) + \int_0^{t \wedge \tau_V} r(Z(s)) dl(s) \]

- \( W = (W(t), t \geq 0) \) is a \( d \)-dimensional Brownian motion with drift vector \( \mu \) and covariance matrix \( A \)
- \( \tau_V := \inf\{ t \geq 0 \mid Z(t) \in \mathcal{V} \} \) is the first hitting moment of the set \( \mathcal{V} \) by the process \( Z \)
- \( l = (l(t), t \geq 0) \) is a continuous real-valued nondecreasing function with \( l(0) = 0 \), which can increase only when \( Z(t) \in \partial D \)
Term with $l$: **hard barrier**, does not allow $Z$ to get out of $\overline{D}$

Same question: can we emulate this by a **soft barrier** created by a drift coefficient?

$$dX(t) = f(X(t))dt + dW(t)$$
First, let us assume that the limiting reflected Brownian motion a.s. does not hit non-smooth parts $\mathcal{V}$ of the boundary $\partial D$.

This is automatically true when the boundary is smooth.

For an important case $\overline{D} = \mathbb{R}^d_+$: multidimensional positive orthant, sufficient conditions are known (Sarantsev, 2015).
Let \( \phi(x) \) be the signed distance from \( x \) to \( \partial D \) (positive if \( x \in D \), negative if \( x \notin \overline{D} \)). Fix a compact subset \( K \subseteq \mathbb{R}^d \setminus \mathcal{V} \). Let

\[
\Delta_n(s) := \min_{\substack{x \in K \\phi(x) = s}} \|f_n(x)\|.
\]

We need: for every \( \varepsilon > 0 \),

\[
\int_{-\varepsilon}^{\varepsilon} \Delta_n(s) \, ds \to \infty.
\]

Then the soft barrier is strong enough to repel \( X_n \) inside \( \overline{D} \) (asymptotically, as \( n \to \infty \)).
We also need the soft barrier to reflect the process in the same direction $r(x)$ when $Z(t) = x$. Let $y_x$ be the closest point to $x$ at the boundary $\partial D$.

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{x \in K, \text{ dist}(x, \partial D) \leq \epsilon} \left\| \frac{f_n(x)}{\|f_n(x)\|} - \frac{r(y_x)}{\|r(y_x)\|} \right\| = 0.$$
Assume these conditions above hold.

In addition, assume $f_n \to 0$ uniformly on compact subsets of $D$.

Then $X_n \Rightarrow Z$ in law on $C([0, T], \mathbb{R}^d)$. 
Weak Convergence
of Obliquely Reflected Diffusions
A reflected diffusion $Z = (Z(t), t \geq 0)$ in the domain $D$.

- When $Z(t) \in D$, it behaves as a solution to an SDE with drift coefficient $g(\cdot)$ and covariance matrix $A(\cdot)$.
- When $Z(t) = x \in \partial D \setminus \mathcal{V}$, it is reflected inside $D$ in the direction of the reflection vector $r(x)$.
- When $Z(t) \in \mathcal{V}$, it is stopped.
Take a sequence \((Z_n)_{n\geq 0}\) of such reflected diffusions.

Assume none hits non-smooth parts of the boundary.

If \(D_n \to D_0, g_n \to g_0, A_n \to A_0, r_n \to r_0\) (in what sense?), then

\[ Z_n \Rightarrow Z_0 \quad \text{in law on} \quad C([0, T], \mathbb{R}^d). \]

This is a generalization of Burdzy & Chen (1998), where they proved this for normally reflected Brownian motions, but under more general conditions.
The functions $g_n$ are defined on different domains $D_n$.

We need: for every sequence $(n_k)$,

\[\text{if } z_{n_k} \in D_{n_k}, \text{ and } z_{n_k} \rightarrow z_0 \in D_0, \text{ then } g_{n_k}(z_{n_k}) \rightarrow g_0(z_0).\]

Same for $A_n$ defined on $D_n$, and $r_n$ defined on $\partial D_n \setminus \mathcal{V}_n$. 
For each $D_n$, let $\phi_n(x)$ be the signed distance to $\partial D_n$: positive inside $D_n$, negative outside.

We say $D_n$ converges weakly to $D_0$ if

$$\phi_n(x) \to \phi_0(x) \quad \text{for all} \quad x \in \mathbb{R}^d.$$ 

This is weaker than Hausdorff convergence, but stronger than Wijsman convergence, which is $\text{dist}(x, D_n) \to \text{dist}(x, D_0)$. 

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Thanks!