We have $N$ Brownian particles on the real line:

$$X_1(t), \ldots, X_N(t).$$

Rank them from bottom to top:

$$X_{(1)}(t) \leq X_{(2)}(t) \leq \ldots \leq X_{(N)}(t).$$

The particle which is currently the $k$th lowest has rank $k$.

**Main Rule:** the particle which currently has rank $k$ moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$.

When particles exchange ranks, they also exchange drift and diffusion coefficients.
Finite Systems of Competing Brownian Particles

When two or more particles are tied, how to assign ranks?

We resolve ties in favor of the lexicographic order: assign lower ranks to particles $X_i$ with smaller indices $i$.

So the SDE governing dynamics of the particles is

$$dX_i(t) = \sum_{k=1}^{N} 1(X_i \text{ has rank } k \text{ at time } t) (g_k dt + \sigma_k dW_i(t)).$$

This system was introduced in (Banner, Fernholz, Karatzas, 2005)
It has a weak solution, unique in law (Bass, Pardoux, 1987)
Motivation

- **Stochastic Portfolio Theory**
  Banner, Fernholz, Karatzas, 2005
  Pal, Chatterjee, 2010
  Jourdain, Reygner, 2013

- **Similar systems**
  Ferrari, Spohn, Weiss, 2015
  Aizenman, Ruzmaikina, 2005
  Aizenman, Arguin, 2009

- **Scaling limits of asymmetrically colliding random walks**
  Karatzas, Pal, Shkolnikov, 2016

- **Discretized version of McKean-Vlasov nonlinear diffusion**
  Shkolnikov, 2012
  Jourdain, Reygner, 2013
  Dembo, Shkolnikov, Varadhan, Zeitouni, 2016
  Kolli, Shkolnikov, 2016
A triple collision:

\[ X_{(k-1)}(t) = X_{(k)}(t) = X_{(k+1)}(t). \]

A simultaneous collision:

\[ X_{(k)}(t) = X_{(k+1)}(t), \quad \text{and} \quad X_{(l)}(t) = X_{(l+1)}(t), \quad k \neq l. \]

A triple collision is a particular case of a simultaneous collision, because we can let \( l = k + 1 \).

This SDE is known to have a strong solution, which is pathwise unique, only up to the first moment of a triple collision (Ichiba, Karatzas, Shkolnikov, 2013). Whether there is a strong solution after this moment is an open question.
Theorem (Ichiba, Karatzas, Shkolnikov, 2013; S, 2015)

There are a.s. no triple and simultaneous collisions if and only if the sequence \( (\sigma_k^2) \) is concave:

\[
\sigma_k^2 \geq \frac{1}{2} \left( \sigma_{k-1}^2 + \sigma_{k+1}^2 \right), \quad k = 2, \ldots, N - 1.
\]

If this condition does not hold for a certain \( k \), then with positive probability there exists a triple collision between particles with ranks \( k - 1, k, k + 1 \).

**Corollary:** If there are a.s. no triple collisions, then there are a.s. no simultaneous collisions. (This was not obvious!)
Fix an $n \geq 4$ and try to avoid collisions of $n$ particles.

**Theorem (Ichiba, S, 2017)**

Assume that for $n \geq 4$, the diffusion coefficients satisfy

$$\max_{1 \leq k \leq N} \sigma_k^2 < \frac{n - 1}{2} \min_{1 \leq k \leq N} \sigma_k^2.$$

Then there are a.s. no collisions of $n$ particles.

Note that this does not depend on $N$, the number of particles.
This is the process taking values in the orthant $\mathbb{R}^{N-1}_+$:

$$Z(t) = (Z_1(t), \ldots, Z_{N-1}(t)),$$

defined as follows: for $k = 1, \ldots, N-1, \ t \geq 0,$

$$Z_k(t) = X_{(k+1)}(t) - X_{(k)}(t).$$
Define \( \bar{g}_k := \frac{1}{k} (g_1 + \ldots + g_k), \quad k = 1, \ldots, N. \)

- There exists a stationary gap distribution \( \pi \) if and only if
  \[
  \bar{g}_k > \bar{g}_N, \quad k = 1, \ldots, N - 1:
  \]
  that is, if the average drift for a few bottom particles is greater than the average drift for all particles.

- If this stationary distribution exists, then it is unique.
  Regardless of the initial condition, \( Z(t) \Rightarrow \pi \) as \( t \to \infty. \)
  Convergence in total variation, exponentially fast.
  To estimate this exponential rate is an open question.

Pal, Pitman, 2008
Banner, Fernholz, Ichiba, Karatzas, Papathanakos, 2011
In case of the following condition:

\[ \sigma_k^2 - \sigma_{k-1}^2 = \sigma_{k+1}^2 - \sigma_k^2, \quad k = 2, \ldots, N - 1, \]

the stationary distribution \( \pi \) is the product of exponentials:

\[
\pi = \bigotimes_{k=1}^{N-1} \text{Exp} \left( \frac{4k(g_k - g_N)}{\sigma_k^2 + \sigma_{k+1}^2} \right).
\]

\( \text{Exp}(\alpha) \) is the exponential distribution with rate \( \alpha \) and mean \( \alpha^{-1} \).

Pal, Pitman, 2008  
Banner, Fernholz, Ichiba, Karatzas, Papathanakos, 2011
\[ g_1 = 1, \ g_2 = \ldots = g_N = 0, \]

\[ \sigma_1 = \sigma_2 = \ldots = \sigma_N = 1. \]

It is called so because the bottom-ranked particle supports all other particles on its shoulders, like the ancient Atlas hero.

It has the following stationary gap distribution:

\[ \pi = \prod_{k=1}^{N-1} \text{Exp} \left( 2 \frac{N - k}{N} \right). \]
Infinitely many Brownian particles on the real line: \( X_1(t), X_2(t), \ldots \) 
Rank them from the bottom upward:

\[
X_{(1)}(t) \leq X_{(2)}(t) \leq X_{(3)}(t) \leq \ldots
\]

Main Rule: the particle which currently has rank \( k \) moves as a Brownian motion with drift \( g_k \) and diffusion \( \sigma_k^2 \). When particles exchange ranks, they also exchange drift and diffusion coefficients.

\[
dX_i(t) = \sum_{k=1}^{\infty} 1(X_i \text{ has rank } k \text{ at time } t) (g_k \, dt + \sigma_k \, dW_i(t)).
\]

Introduced in (Shkolnikov, 2011), studied in (Ichiba, Karatzas, Shkolnikov, 2013; S, 2016; Dembo, Tsai, 2015; S, Tsai, 2016).
Unlike the finite system, the infinite system might not always exist, even in the weak sense.

**Why?** Because it might not be possible to rank these particles.

Assume for simplicity that \( g_1 = g_2 = \ldots = 0 \) and \( \sigma_1 = \sigma_2 = \ldots = 1 \). Start infinitely many i.i.d. Brownian motions from the same initial point. Then it is impossible to rank them from bottom to top.

This shows that we need to have some condition on initial values \( x_n = X_n(0), \ n = 1, 2, \ldots \) They should be far enough apart, or, alternatively, tend to infinity \( x_n \to \infty \) fast enough.
An assumption on initial conditions $x_n = X_n(0), \ n = 1, 2, \ldots$

Tending to infinity fast enough?

$$\sum_{n=1}^{\infty} \exp(-\alpha x_n^2) < \infty \text{ for all } \alpha > 0.$$ 

E.g. $x_n = cn, \ x_n = c \log n$ work, but $x_n = c (\log n)^{1/2}$ does not.

**Theorem (Ichiba, Karatzas, Shkolnikov, 2013; S, 2016)**

*Suppose $X_n(0) = x_n \to \infty$ fast enough. In addition,*

(a) *for some $n_0$, $g_{n_0} = g_{n_0+1} = \ldots$ and $\sigma_{n_0} = \sigma_{n_0+1} = \ldots$, or:*

(b) $\sigma_n = 1$ for all $n$, and $\sum_{n=1}^{\infty} g_n^2 < \infty$.

*Then the system exists in the weak sense and is unique in law.*
The gap process for infinite systems $X_1(t), X_2(t), \ldots$ is defined similarly to the finite case: it is an $\mathbb{R}_+^\infty$-valued process

$$Z(t) = (Z_1(t), Z_2(t), Z_3(t), \ldots), \quad t \geq 0,$$

$$Z_k(t) = X_{(k+1)}(t) - X_{(k)}(t),$$

where $X_{(k)}$ is the $k$th ranked particle.

We are interested in stationary gap distributions $\pi$ and weak convergence $Z(t) \Rightarrow \pi$ as $t \to \infty$. 
Consider the case

\[ g_1, \ldots, g_M > 0, \quad g_{M+1} = g_{M+2} = \ldots = 0, \]

\[ \sigma_1 = \sigma_2 = \ldots = 1. \]

The results are available in a more general setting, but here they are simple. A particular case is the infinite Atlas model:

\[ g_1 = 1, \quad g_2 = g_3 = \ldots = 0; \quad \sigma_1 = \sigma_2 = \ldots = 1. \]
Find stationary gap distribution $\pi$ for infinite Atlas model. Stationary gap distribution for Atlas model with $N$ particles is

$$\pi_N = \bigotimes_{k=1}^{N-1} \text{Exp} \left( 2 \frac{N - k}{N} \right).$$

For each $k$, letting $N \to \infty$, we get:

$$2 \frac{N - k}{N} \to 2.$$

It is reasonable to conjecture that $\pi = \bigotimes_{k=1}^{\infty} \text{Exp}(2)$. This was formally proved in Pal, Pitman, 2008.
Theorem (S, Tsai, 2016)

There are infinitely many stationary gap distributions:

\[ \pi_a = \bigotimes_{k=1}^{\infty} \text{Exp}(2(g_1 + \ldots + g_k) + ak), \ a \geq 0. \]

In particular, for the infinite Atlas model we have:

\[ \pi_a = \bigotimes_{k=1}^{\infty} \text{Exp}(2 + ka), \ a \geq 0. \]

This includes the result from (Pal, Pitman, 2008):

\[ \pi_0 = \bigotimes_{k=1}^{\infty} \text{Exp}(2). \]
Consider again the stationary gap distribution:

\[ \pi_a = \bigotimes_{k=1}^{\infty} \text{Exp}(2(g_1 + \ldots + g_k) + ak), \quad a \geq 0. \]

For \( a > 0 \), there are \( O(\exp(aL)) \) particles on \([0, L]\), as \( L \to \infty \).

The quantity of particles is growing exponentially fast when you go to infinity. This creates a negative drift:

\[ \mathbb{E} \left[ X_{(k)}(t) - X_{(k)}(0) \right] = -\frac{a}{2} t. \]

The sheer density of particles at the top pushes the bottom particles down with (on average) linear speed.
Two probability measures $P$ and $Q$ on $\mathbb{R}_+^\infty$. We say:

$$P \preceq Q : Q \text{ stochastically dominates } P,$$

if for all $x_1, x_2, \ldots \geq 0$,

$$P\left( [x_1, \infty) \times [x_2, \infty) \times \ldots \right) \leq Q\left( [x_1, \infty) \times [x_2, \infty) \times \ldots \right).$$
Recall: for finite systems, the gap process always converges to its stationary distribution as $t \to \infty$.

For infinite systems, this is no longer true, because there are multiple stationary gap distributions and therefore multiple domains of convergence.

Among stationary gap distributions, this one plays a special role:

$$
\pi_0 = \bigotimes_{k=1}^{\infty} \text{Exp}(2(g_1 + \ldots + g_k)).
$$

For the infinite Atlas model, this takes the form

$$
\pi_0 = \bigotimes_{k=1}^{\infty} \text{Exp}(2).
$$
Theorem (S, 2016)

(a) The family \((Z(t), t \geq 0)\) of random variables is tight in \(\mathbb{R}^\infty\).

(b) Every weak limit point of \(Z(t)\) as \(t \to \infty\) is stochastically smaller than the stationary distribution \(\pi_0\).

(c) If \(Z(0)\) is stochastically larger than \(\pi_0\), then \(Z(t) \Rightarrow \pi_0\) as \(t \to \infty\).

Part (c) describes (part of) the domain of convergence for the distribution \(\pi_0\). Other domains of convergence are not known.
Infinitely many driftless Brownian particles $X_1(t), X_2(t), \ldots$

$$g_1 = g_2 = \ldots = 0, \quad \sigma_1 = \sigma_2 = \ldots = 1.$$ 

(S, Tsai, 2016) A family of stationary gap distributions:

$$\pi_{a} = \bigotimes_{k=1}^{\infty} \text{Exp}(ka), \quad a > 0.$$ 

Consistent with (Ruzmaikina, Aizenman, 2005), which is a discrete-time version of the model. There is an implied drift:

$$\mathbb{E} \left[ X_{(k)}(t) - X_{(k)}(0) \right] = -\frac{a}{2} t.$$ 

Even without any original drifts, the sheer density of particles is enough to create new drift and a stationary gap distribution.
Infinite system with negative drift at the bottom:

\[ g_1 = -1, \ g_2 = g_3 = \ldots = 0, \ \sigma_1 = \sigma_2 = \ldots = 1. \]

(S, Tsai, 2016) A family of stationary gap distributions:

\[ \pi_a = \bigotimes_{k=1}^{\infty} \text{Exp}(-2 + ka), \quad a > 2. \]

Even with negative bottom drift, if the exponential density of particles above is large enough, it can outweigh this drift.

The particles above chase the bottom particle fast enough.
(Dembo, Tsai, 2015) Consider the infinite Atlas model

\[ g_1 = 1, \ g_2 = g_3 = \ldots = 0, \ \sigma_1^2 = \sigma_2^2 = \ldots = 1 \]

in its stationary gap distribution \( \pi_0 := \prod_{k=1}^{\infty} \text{Exp}(2) \).

**Theorem (Dembo, Tsai, 2015)**

\[
\left( \varepsilon^{1/4} X_{(1)}(t/\varepsilon), \ t \geq 0 \right) \Rightarrow cB_{1/4}, \ \varepsilon \to 0.
\]

\( B_{1/4} \) is the fractional Brownian motion with \( \mathbb{E}B_{1/4}(t)^2 = t^{1/2} \).

As \( t \to \infty \), \( X_{(1)}(t) \) behaves as \( B_{1/4} \), not a Brownian motion.

Not a Brownian scaling.
Competing Lévy Particles
Rank-based systems driven by Lévy processes rather than Brownian motions. (Shkolnikov, 2011; S, 2016)

Hybrid Models
Drift and diffusion coefficients depending on ranks and names of particles. (Banner, Fernholz, Ichiba, Karatzas, Papathanakos, 2011; Fernholz, Ichiba, Karatzas, 2013)

Two-Sided Infinite Systems
Particles $X_n(t), \ n \in \mathbb{Z}$, with $\ldots \leq X_{-1} \leq X_0 \leq X_1 \leq \ldots$ (Harris, 1965; S, 2016)

Asymmetric Collisions
Particles having different mass and moving away with different speed when colliding. (Karatzas, Pal, Shkolnikov, 2016)
Thanks!