SOLUTIONS

STAT 706 EXAM 1 - SPRING 2018

INSTRUCTIONS: Write your solutions concisely, and in the space provided. Please show your work or otherwise justify your claims to receive full credit. I cannot award partial credit for blank answers! The exam is worth 100 points total.

1. (15 pts) State the definition of a probability space \((\Omega, \mathcal{F}, P)\), including the definition and key properties of each part of the probability triple.

-+3\ 
\[ \Omega = \text{sample space (a non-empty set)} \]

-+6\ 
\[ \mathcal{F} = \sigma\text{-algebra: collection of subsets of } \Omega \text{ containing } \emptyset, \Omega, A \subseteq \Omega ; \text{closed under complements, countable unions (and countable intersections).} \]

++6\ 
\[ P = \text{probability measure: function from } \mathcal{F} \text{ to } [0,1] \text{ with } P(\emptyset) = 0, P(\Omega) = 1, 0 \leq P(A) \leq 1 \text{ for all } A \in \mathcal{F}, \text{ and s.t. } P \text{ is countably additive.} \]

2. (5 pts) Give an example of a probability space \((\Omega, \mathcal{F}, P)\).

\[ \Omega = \{1, 2, 3\} \]
\[ \mathcal{F} = 2^\Omega = \{ \emptyset, \{1\}, \{1, 2, 3\}, \{1, 2\}, \{1\}, \{2\}, \{3\}, \Omega \} \]
\[ P(\emptyset) = 0, \ P(\Omega) = 1, \ P(\{1\}) = P(\{2, 3\}) = \frac{1}{2} \]
3. (28 pts) Given a sample space $\Omega$, determine whether the given collection of subsets $\mathcal{F}$ is an algebra, a $\sigma$-algebra, or neither. Justify your answers.

(a) $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = \{\emptyset, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$. 

(i) $\emptyset, \Omega \in \mathcal{F}$ 

(ii) For $A \in \mathcal{F}$, $A^c \in \mathcal{F}$ (closed under complementation) 

(iii) For $A_1, A_2, \ldots \in \mathcal{F}$, 

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$ (closed under countable unions)

Thus, $\mathcal{F}$ is a $\sigma$-algebra (and also an algebra).

(b) $\Omega = \mathbb{Z}$ and $\mathcal{F} = \{A \subset \Omega : A$ is finite\}. 

Let $A \in \mathcal{F}$. Then $A$ is finite, say 

$A = \{1, 2, 3\}$. But $A^c = \mathbb{Z} \setminus A = \mathbb{Z} \setminus \{1, 2, 3\}$

which is an infinite set, hence $A^c \notin \mathcal{F}$. 

Since $\mathcal{F}$ is not closed under complementation, 

it is neither an algebra nor a $\sigma$-algebra.

[Also, $\emptyset \in \mathcal{F}$ but $\Omega \notin \mathcal{F}$ since $\Omega = \mathbb{Z}$ is infinite.]
\[ A_1, A_2 \in \mathcal{T}, \ A_1 \text{ infinite } \Rightarrow A_1^c \text{ finite.} \]

Let \( A = A_1 \cup A_2 \). Show \( A \in \mathcal{T} \). Note that \( A^c = (A_1 \cup A_2)^c = A_1^c \cap A_2^c \subseteq A_1^c \).

(c) \( \Omega = \mathbb{Z} \) and \( \mathcal{F} = \{ A \subseteq \Omega : \text{either } A \text{ or } A^c \text{ is finite} \} \).

\[ \text{algebra} \]

(i) \( \emptyset, \Omega \in \mathcal{T} \).

(ii) Let \( A \in \mathcal{T} \) and let \( B = A^c \). Then

\[ \begin{align*}
A \text{ finite } & \Rightarrow B^c \text{ finite (since } B^c = (A^c)^c = A) \\
A^c \text{ finite } & \Rightarrow B \text{ finite (since } B = A^c) 
\end{align*} \]

Either way, \( B \in \mathcal{T} \). \( \mathcal{T} \) is closed under complementation.

(iii) Let \( A_1, \ldots, A_n \in \mathcal{T} \) and let \( A = \bigcup_{i=1}^n A_i \) (finite union).

If all \( A_i \)'s finite, then \( A \) is finite. If at least 1 \( A_i \) is infinite (wlog say \( A_1 \)), then \( A^c \subseteq A_1^c \) so \( A^c \) is finite.

Either way, \( A \in \mathcal{T} \). (See * above)

To see that \( \mathcal{T} \) is not a \( \sigma \)-algebra, note that \( \{ 2n \} \in \mathcal{T} \) for each \( n \in \mathbb{Z} \) but \( \bigcup_{n \in \mathbb{Z}} \{ 2n \} = \text{set of even integers} \notin \mathcal{T} \) since its complement is also infinite.

(d) What is the difference between an algebra and a \( \sigma \)-algebra?

The only difference is that an algebra is only closed under finite unions, whereas a \( \sigma \)-algebra is closed under countable unions.
4. (22 pts) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $B \in \mathcal{F}$ be an event such that $P(B) > 0$.

(a) Show that $Q(A) = \frac{P(A \cap B)}{P(B)}$ for $A \in \mathcal{F}$ also defines a probability measure on $(\Omega, \mathcal{F})$.

Let $B \in \mathcal{F}$ s.t. $P(B) > 0$.

$$Q(\emptyset) = \frac{P(\emptyset \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0.$$

$$Q(\Omega) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

**Countable Additivity:** Let $A_1, A_2, \ldots$ be pairwise disjoint sets in $\mathcal{F}$. Then

$$Q\left(\bigcup_{i=1}^{\infty} A_i\right) = \frac{P\left(\bigcup_{i=1}^{\infty} A_i \cap B\right)}{P(B)}$$

$$= \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$

since $P$ is countably additive

$$= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} Q(A_i).$$

Also, $P(A \cap B) \leq P(B)$ since $A \cap B \subseteq B \ \forall \ A \in \mathcal{F}$.

$$\Rightarrow Q(A) \leq 1 \ \forall \ A \in \mathcal{F}.$$

Thus, $Q$ also defines a probability measure on $(\Omega, \mathcal{F})$.
(b) Show that \( Q(A) = P(A) \) if and only if \( A \) and \( B \) are independent events with respect to \( P \).

Suppose that \( Q(A) = P(A) \) \( \forall A \in \mathcal{F} \). Then

\[
Q(A) = \frac{P(A \cap B)}{P(B)} = P(A)
\]

\[\Rightarrow P(A \cap B) = P(A)P(B).\]

Thus, events \( A \) and \( B \) are independent w.r.t. \( P \).

Suppose that events \( A \) and \( B \) are independent w.r.t. \( P \).

Then \( P(A \cap B) = P(A)P(B) \) by defn

\[\Rightarrow P(A) = \frac{P(A \cap B)}{P(B)} \quad \text{for all } B \in \mathcal{F} \text{ s.t. } P(B) > 0\]

\[\Rightarrow P(A) = Q(A) \quad \forall A \in \mathcal{F}.\]
5. (9 pts) True or False? Justify your answers. For a sequence of events \(A_1, A_2, \ldots\),

(a) \(\limsup_n A_n = \{A_n \text{ i.o.}\}\). True, by definition \(\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\).

(b) \(\limsup_n A_n \subseteq \liminf_n A_n\). False, \(\liminf_n A_n \subseteq \limsup_n A_n\).

(c) If \(\{A_n\}\) are independent, then \(P(\limsup_n A_n)\) is either 0 or 1. True, follows from Borel-Cantelli Lemmas I \& II.

6. (11 pts) Let \(\{X_n\}_{n=1}^{\infty}\) be independent random variables with \(X_n \sim \text{Uniform}(\{1, 2, \ldots, n\})\). That is, \(P(A) = |A|/|\Omega|\) for all \(A \in \mathcal{F} = 2^\Omega\) (where \(|A|\) is the cardinality of set \(A \subseteq \Omega\)).
Compute \(P(X_n = 5 \text{ i.o.})\), the probability that an infinite number of the \(X_n\) are equal to 5.

\[
P(X_n = 5) = \left| \frac{\left| \{5\} \right|}{|\Omega|} \right| = \frac{1}{n} \quad \text{for } n \geq 5.
\]

\[
\sum_{n=1}^{\infty} P(X_n = 5) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \quad \text{Note that the}
\]

\(X_n\)'s are independent (by assumption). Then by the Borel-Cantelli lemma, it follows that

\[
P(X_n = 5 \text{ i.o.}) = 1.
\]
7. (10 pts) Let \((\Omega, \mathcal{F}, P)\) be Lebesgue measure on \(\Omega = [0, 1]\) and define simple random variables \(X\) and \(Y\) by

\[
X(\omega) = \begin{cases} 
1, & \text{if } \omega > 1/3 \\
3, & \text{if } \omega \leq 1/3 
\end{cases}
\]

\[
Y(\omega) = \begin{cases} 
2, & \text{if } \omega \in \mathbb{Q} \\
4, & \text{if } \omega = 1/\sqrt{2} \\
6, & \text{if other } \omega \leq 1/4 \\
8, & \text{otherwise}
\end{cases}
\]

Compute the expected values of \(X\) and \(Y\).

\[
E[X] = 1 \cdot \lambda((\frac{1}{3}, 1]) + 3 \cdot \lambda([0, \frac{1}{3}])
\]

\[
= \frac{2}{3} + 3 \cdot \frac{1}{3} = \frac{5}{3}
\]

\[
E[Y] = 2 \lambda(\mathbb{Q} \cap [0, 1]) + 4 \lambda([\frac{1}{\sqrt{2}}, 1]) + 6 \lambda([0, \frac{1}{4}] \setminus \mathbb{Q} \cap [0, 1]) + 8 \lambda((\frac{1}{4}, 1] \setminus \mathbb{Q} \cup [0, 1])
\]

\[
= 6 \cdot \frac{1}{4} + 8 \cdot \frac{3}{4}
\]

\[
= \frac{3}{2} + \frac{12}{2} = \frac{15}{2}
\]