Given any Borel-measurable function \( f \) (aka density fct) st. \( f \geq 0 \) and \( \int_{-\infty}^{\infty} f(t) \, d\lambda(t) = 1 \), we can define a distribution function (law) \( \mu \) by:

\[
\mu(B) = \int_{-\infty}^{\infty} f(t) \, 1_B(t) \, d\lambda(t), \quad B \in B \quad (\text{Borel set})
\]

[Sometimes write this as \( \mu(B) = \int_B f \, d\lambda \) or \( \int_B f(t) \, d\lambda(t) \).]

\[
\int_B d\mu(t) = \int_B f(t) \, d\lambda(t) \quad \forall B \in B \quad \Leftrightarrow \quad d\mu(t) = f(t) \, d\lambda(t)
\]

\[
\frac{d\mu}{d\lambda} = f
\]

"with respect to" \( \lambda \)

\( \mu \) is absolutely continuous w.r.t. \( \lambda \)

\( f \) is the density for \( \mu \) w.r.t. \( \lambda \)

Prop 6.2.3: Suppose \( \mu \) has density \( f \) w.r.t. \( \lambda \). Then for any Borel-measurable function \( g: \mathbb{R} \to \mathbb{R} \),

\[
E_{\mu}[g] = \int_{-\infty}^{\infty} g(t) \, d\mu(t) = \int_{-\infty}^{\infty} g(t) \, f(t) \, d\lambda(t)
\]

provided either side is well-defined.
Now it is possible to do explicit computations with absolutely continuous RVs:

\[ X \sim \mathcal{N}(0, 1) \]

\[
E[X] = \int t \, d\mu_X(t) = \int t \, \phi(t) \, d\lambda(t)
\]

\[
= \int_{-\infty}^{\infty} t \, \phi(t) \, dt
\]

\[
= \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt
\]

More generally,

\[
E[g(X)] = \int g(t) \, d\mu_X(t) = \int g(t) \, \phi(t) \, d\lambda(t)
\]

\[
= \int_{-\infty}^{\infty} g(t) \, \phi(t) \, dt
\]

for any

Riemann-integrable function \( g \)
Stochastic Processes

(Ref: §7 Rosenthal, §7 Billingsley)

def: A discrete time stochastic process is a sequence of random variables $X_0, X_1, X_2, \ldots$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Typically, the $X_n$'s are not independent. Think of the index "n" as representing time.

$\Rightarrow X_n$ is the value of a random quantity at time $n$.

Example: Infinite fair coin tossing $(r_1, r_2, \ldots)$

where $r_i = 0$ or $1$ with prob. $\frac{1}{2}$. Say 0 = Tails, 1 = Heads.

Set $X_0 = 0$; $X_n = r_1 + \ldots + r_n$, $n \geq 1$

$\Rightarrow$ # of heads obtained by time $n$

Thm [Existence]: Let $\mu_1, \mu_2, \ldots$ be any sequence of Borel probability measures on $\mathbb{R}$. Then there exists a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ and RVs $X_1, X_2, \ldots$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $X_n$ are independent and $\mathbb{E}(X_n) = \mu_n$. 
Gambling & Gambler's Ruin

Let $Z_1, Z_2, \ldots$ be i.i.d. RVs s.t.

$$P(Z_i = 1) = p \quad \text{and} \quad P(Z_i = -1) = 1 - p \quad \text{for some fixed } p, 0 < p < 1.$$

Let $X_n = a + Z_1 + \ldots + Z_n$ with $X_0 = a$ for some $a \in \mathbb{N}$.

Interpret $X_n$ as gambler's fortune (in $\$)$ at time $n$ when repeatedly making $\$1$ bets.

Then $\{X_n\}$ is a stochastic process known as a simple random walk.

- Player begins with $\$a$
- At each time step, either wins $\$1$ w/prob $p$
  or loses $\$1$ w/prob. $1-p$.

Q. What is the distribution of $X_n$?

$$P(X_n = a + k) = 0 \quad \text{unless } -n \leq k \leq n$$

with $n + k$ even

Say $a = 2$.

$$X_1 = 2 + Z_1$$

$\begin{cases} 2 + 1 & \text{if } Z_1 = 1 \quad n + k = 1 + 1 \text{ (even)} \\ 2 - 1 & \text{if } Z_1 = -1 \quad n + k = 1 - 1 \text{ (even)} \end{cases}$

$$X_2 = 2 + Z_1 + Z_2$$

$K = -2, 0, 2 \quad n = 2$ so $n + k$ is even
For such \( k \), there are \( \binom{n}{n+k} \) different possible sequences \( Z_1, Z_2, \ldots \) s.t. \( X_n = a + k \), i.e. all seqs consisting of \( \frac{n+k}{2} \) 1's and \( \frac{n-k}{2} \) -1's.

\[
\begin{align*}
\text{e.g. } X_2 &= 2 + Z_1 + Z_2 \\
-1 &-1 \\
-1 &+1 \\
+1 &-1 \\
+1 &+1
\end{align*}
\]

\( n+k = 2+2 = 4 \)
\( \frac{n+k}{2} = \frac{4}{2} = 2 \) 1's
\( \frac{n-k}{2} = \frac{0}{2} = 0 \) -1's

Each such sequence has probability \( p^{\frac{n+k}{2}}(1-p)^{\frac{n-k}{2}} \)

Thus,

\[
P(X_n = a + k) = \binom{n}{n+k} p^{\frac{n+k}{2}}(1-p)^{\frac{n-k}{2}}, \quad -n \leq k \leq n \text{ \( \frac{1}{2} \) } \quad n+k \text{ even}
\]

\( \frac{1}{2} 0 \) otherwise.

Gambler's Ruin Problem:

Suppose that \( 0 < a < c \), and let

\( \tau_0 = \inf\{n > 0 : X_n = 0\} \) initial capital

\( \tau_c = \inf\{n > 0 : X_n = c\} \) be the first hitting time of \( 0 \) and \( c \), respectively.
Gambler's ruin question is: what is $P(T_c < T_o)$?

In words, what is the probability the gambler will get rich (reach #c) before going broke (reach #0)?

(Note: $\exists T_c < T_o \iff$ includes the case $T_o = \infty$ while $T_c < \infty$)

but not the case $T_c = T_o = \infty$.

Solving this is not straightforward.

There's a nice trick!

Set $s(a) = P(T_c < T_o)$.

Write dependence on $a$ explicitly allows us to vary $a$ and relate $s(a)$ to $s(a-1)$ or $s(a+1)$

For $1 \leq a \leq c-1$, we have

$s(a) = P(T_c < T_o)$

$= P(Z_1 = -1, T_c < T_o) + P(Z_1 = 1, T_c < T_o)$

$= (1-p) s(a-1) + p s(a+1)$

since $Z_c$'s indep. of $Z_1, Z_2, ..., Z_c$ is a probabilistic replica of $Z_1, Z_2, ...$

Further, $s(0) = 0$ & $s(c) = 1$ by definition

If $a = 0$, can never play

If $a = c$, clearly you reach #c before #0.
Solve system of \( c-1 \) equations for the \( c-1 \) unknowns:

\[ s(1), s(2), \ldots, s(c-1) \quad \text{\textasciitilde HW problem} \]

To obtain

\[
\begin{cases} 
  s(a) = P(\tau_c < \tau_0) = \frac{1 - \left( \frac{a}{p} \right)^a}{1 - \left( \frac{a}{p} \right)^c} & \text{for } p \neq \frac{1}{2} \\
  s(a) = \frac{a}{c} & \text{for } p = \frac{1}{2}
\end{cases}
\]

\text{Note:} \quad (q = 1 - p)

Specific Example:

Suppose you start with $9,700 \ (a = 9,700) \ and \ your \ goal \ is \ to \ win \ $10,000 \ before \ going \ broke \ (c = 10,000).

If \( p = \frac{1}{2} \), then prob. of success = \( \frac{a}{c} = 0.97 \)

\[ P(\tau_c < \tau_0) \quad \text{\textasciitilde very high!} \]

If \( p = 0.49 < \frac{1}{2} \), then prob. of success is

\[
1 - \left( \frac{0.51}{0.49} \right)^{9700} \approx 0.0000061 \quad \text{or} \quad 1 \text{ chance in } 163,000.
\]

\text{dramatic change!} \quad \text{only small disadv. on each bet...}
Q. What is the probability the gambler will go broke if they never stop gambling? i.e. \( P(\tau_0 < \infty) \)

Let \( H_c = \{ \tau_0 < \tau_c \} \). Then \( \{ H_c \} \) is increasing up to \( \{ \tau_0 < \infty \} \) as \( c \to \infty \). So by continuity of prob.,

\[
P(\tau_0 < \infty) = \lim_{c \to \infty} P(H_c)
\]

\[
= \lim_{c \to \infty} r(a)
\]

where \( r(a) = P(\tau_0 < \tau_c) \)

\[
\begin{cases}
1 & \text{if } p \leq \frac{1}{2} \\
\left( \frac{q}{p} \right)^a & \text{if } p > \frac{1}{2}
\end{cases}
\]

(Also note that \( r(a) = 5(c-a) \))

Thus, if \( p \leq \frac{1}{2} \), the gambler is certain to go broke!

Hence, "Gambler's ruin".

Only chance for success occurs if \( p > \frac{1}{2} \).

---

Suppose now that the gambler is allowed to choose how much to bet each time. That is, choose RVs \( W_n \) s.t. fortune at time \( n \) is

\[
X_n = a + W_1Z_1 + \cdots + W_nZ_n
\]

with \( \{ Z_n \} \) as before, \( \frac{1}{2} \) \( W_n \geq 0 \)
Note that \( X_n = X_{n+1} + W_n Z_n \), \( W_n \) \& \( Z_n \) are indep.

and \( E[Z_n] = p - q \).

\[ \Rightarrow E[X_n] = E[X_{n-1}] + (p-q)E[W_n] \]

Then

(i) If \( p = \frac{1}{2} \), \( E[X_n] = E[X_{n-1}] = \cdots = E[X_0] = a \)

and \( \lim_{n \to \infty} E[X_n] = a \).

(ii) If \( p \leq \frac{1}{2} \), \( E[X_n] \leq E[X_{n-1}] \leq \cdots \leq E[X_0] = a \)

\( \text{(since } W_n > 0 \Rightarrow E[W_n] > 0 \) \)

and \( \lim_{n \to \infty} E[X_n] \leq a \).

(iii) If \( p > \frac{1}{2} \), \( E[X_n] \geq E[X_{n-1}] \geq \cdots \geq E[X_0] = a \)

and \( \lim_{n \to \infty} E[X_n] \geq a \).

With "clever betting" (e.g. double until you win), a gambler can cheat fate. However, this requires infinite capital.

(always win)

If capital is finite (as in reality) and \( p \leq \frac{1}{2} \), then on average they will always lose by the Bounded Convergence Thm.
Thm [Bounded Convergence]: Let \( \{X_n\} \) be a sequence of RVs with \( \lim_{n \to \infty} X_n = X \). Suppose \( \exists K \in \mathbb{R} \) s.t.
\[
|X_n| \leq K \quad \forall n \in \mathbb{N} \quad \text{(i.e. } X_n \text{ are uniformly bounded).}
\]
Then \( E[X] = \lim_{n \to \infty} E[X_n] \).