Generalizations of Poisson Process (con't)

Cox Process - a doubly stochastic Poisson process

\[ \lambda = \{ \lambda(t) : t \geq 0 \} \leftarrow \text{rate } \lambda \text{ is also a stochastic process!} \]

Such processes are used to model spike trains (the sequence of action potentials generated by a neuron) and also in financial math to model prices of financial instruments in which credit risk is a significant factor.

(Possible Project Idea!)
Continuous-time Markov Chains
[Ref: Models ch. 6]

- We've already discussed one example: Poisson Process.
  Let $N(t) =$ total # of arrivals by time $t$
  be the state of the process at time $t$.
  Then $\{N(t) : t \geq 0\}$ is a continuous-time MC
  $\mathcal{S} = \{0, 1, 2, \ldots\}$
  Transitions only occur from state $n$ to $n+1$ $(n \geq 0)$
  This type of process is also called a
  Pure Birth Process (or Yule Process).

- If we allow transitions from $n$ to $n-1$ as well as
  from $n$ to $n+1$, then we have a
  Birth & Death Process

  def: The process $\{X(t) : t \geq 0\}$ is a continuous-time MC if
  $P(X(t+s) = j \mid X(s) = i, X(u) = x(u) \text{ for } 0 \leq u < s)
  = P(X(t+s) = j \mid X(s) = i) \quad \wedge \text{Markov Property}$
\[ X(t) = \text{state at time } t \text{ (where } t \text{ is continuous) } \]

The continuous-time MC has stationary transition probabilities if \[ P(X(t+s) = j \mid X(s) = i) \] is independent of \( s \).

\[ \begin{array}{c}
0 & s & s+t \\
\text{time} & & \\
\text{length } t & & \text{length } t
\end{array} \]

- \( T_i \) = amount of time that the process stays in state \( i \) before transitioning into a different state.

\[ T_i \sim \text{exponential } (\nu_i) \Rightarrow \text{"waiting time" is exponentially distributed with rate } \nu_i \]

\[ \Rightarrow \text{average time until a transition is } \frac{1}{\nu_i} \]

- When the process leaves state \( i \), it transitions to state \( j \) with probability \( P_{ij} \)

\[ \text{(Must satisfy } P_{ii} = 0 \text{ and } \sum_j P_{ij} = 1 \text{)} \]

- The amount of time spent in state \( i \) & the next state visited must be independent. Why?

\[ \text{(Info about how long the process has already been in state } i \text{ would be relevant to the prediction of the next state \( \rightarrow \) contradicts MP)} \]
In other words, a \textit{continuous-time MC} is a stochastic process that moves from state to state according to a \textit{discrete-time MC}, but hangs out in each state for an exponential amount of time before transitioning. "embedded Markov chain"

**Example: Shoe Shine Shop**

2 chairs (chair 1 & chair 2) \textsuperscript{a} 2 stage process

A customer goes to chair 1 first $\rightarrow$ shoes are cleaned $\frac{1}{2}$ polish applies

The customer moves to chair 2 next $\rightarrow$ polish is buffed

Service times at the 2 chairs are assumed to be independent exponential RVs with rates $\mu_1$ & $\mu_2$, resp.

Suppose that potential customers arrive according to a Poisson process with rate $\lambda$, and that a customer only enters the system if both chairs are empty.

\# of customers in the system: 0 or 1

(But, if \exists a customer in system, we need to)

Know what chair he/she is in
State Space of CTMC: 0, 1, 2

<table>
<thead>
<tr>
<th>State</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>system is empty</td>
</tr>
<tr>
<td>1</td>
<td>a customer is in chair 1</td>
</tr>
<tr>
<td>2</td>
<td>a customer is in chair 2</td>
</tr>
</tbody>
</table>

Then transition probabilities

\[ P_{01} = P_{12} = P_{20} = 1 \quad (\text{all others are } 0) \]

And rates \( \nu_i \) (rate at which the process makes a transition)
when in state \( i \)

\[ \begin{align*}
\nu_0 &= \lambda & \text{customers arrive, Poisson process} \\
\nu_1 &= \mu_1 & \text{exponential(\( \mu_1 \)) service time in chair 1} \\
\nu_2 &= \mu_2 & \text{exponential(\( \mu_2 \)) in chair 2}
\end{align*} \]

Instantaneous Transition Rates \( q_{ij} \)

For any pair of states \( i \neq j \), let

\[ q_{ij} = \nu_i P_{ij} \]

\( \nu_i \) is the rate defined above

\( P_{ij} \) is the probability that this transition is into state \( j \)

\( \Rightarrow q_{ij} \) is the rate, when in state \( i \), at which the process makes
Then \( v_i = \sum_j v_i \ p_{ij} = \sum_j q_{ij} \) and (since rows of \( P \) sum to 1)

\[
P_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_j q_{ij}}.
\]

Thus, specifying \( q_{ij} \) determines the parameters of the continuous-time Markov chain.

Birth and Death Process

System whose state at any time = # of people in the system at that time.

Suppose that whenever there are \( i \) people in the system,

- new arrivals ("births") enter the system at an exponential rate \( \lambda_i \)
- people leave the system ("deaths") at an exponential rate \( \mu_i \)

Such a system is a birth and death process.

\[ \{ \lambda_i \}_{i=0}^\infty \text{ - birth rates} \]
\[ \{ \mu_i \}_{i=0}^\infty \text{ - death rates} \]