Cor 8.4.6: If a MC has no positive recurrent states, then it does not have a stationary dist’n. 
(all \( \pi_j = 0 \) which contradicts \( \sum_j \pi_j = 1 \))

Cor 8.4.7: Let \( i \neq j \) be 2 states of a MC that communicate (i.e. \( f_{ij} > 0 \) and \( f_{ji} > 0 \)). If \( i \) is positive recurrent then so is \( j \).

Cor 8.4.8: For an irreducible MC (i.e. all states communicate), either all states are positive recurrent or none are.

Thm 8.4.9: For an irreducible MC, either

(a) All states are positive recurrent, \( \exists \) a unique stationary dist’n given by \( \pi_j = \frac{1}{m_j} \) 
(if also aperiodic, \( P_i(X_n=j) \to \pi_j \quad \text{as} \quad n \to \infty \)), OR

(b) No states are pos. recurrent, and \( \not\exists \) a stationary distribution.

Example 1: Symmetric simple RW on \( \mathbb{Z} \) is null recurrent, falls into category (b).

Example 2: MC on finite \( S \) necessarily falls into category (a).
Prop 8.4.10: For an irreducible MC in a finite state space, all states are positive recurrent (i.e., hence a unique stat. dist'n exists).

PF: Fix state \( i \in S \). Let \( h_{ji}^{(m)} = P_j(X_k = i \text{ for some } 1 \leq k \leq m) \)

\[
= \sum_{n=1}^{m} \xi^{(n)}_{ji}.
\]

Then
\[
\lim_{m \to \infty} h_{ji}^{(m)} = \lim_{m \to \infty} \sum_{n=1}^{m} \xi^{(n)}_{ji} = f_{ji} \quad \text{(by def)}
\]

by irreducibility. (for each \( j \in S \))

Since \( S \) is finite, we can find \( m \in \mathbb{N} \) and \( \delta > 0 \) s.t.
\( h_{ji}^{(m)} \geq \delta \) \( \forall j \).

We must also have
\[
1 - h_{ii}^{(n)} \leq (1 - \delta)^{L_{ii}^{(m)}}
\]
so that

letting \( \tau_i = \inf\{n \geq 1 : X_n = i\} \), we have that

\[
m_i = \sum_{n=0}^{\infty} P_i(\tau_i \geq n+1) = \sum_{n=0}^{\infty} (1 - h_{ii}^{(n)})
\]

\[
\leq \sum_{n=0}^{\infty} (1 - \delta)^{L_{ii}^{(m)}}
\]

\[
= \frac{m}{\delta} < \infty.
\]

Thus, all states are pos. recurrent.
Limit Theorems

Ref: R section 9, B section 16.

Some results needed for more advanced topics to come.

Suppose $X_1, X_2, \ldots$ are random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Also, suppose that

$$\lim_{n \to \infty} X_n(w) = X(w) \quad \text{for all } w \in \Omega \text{ outside a set of probability 0.}$$

In other words,

$$X_n \xrightarrow{a.s.} X$$

Q. Does it follow that $\lim_{n \to \infty} E[X_n] = E[X]$?

No, not true in general.

Simple counter example:

$$\Omega = \mathbb{N}$$

$$P(\omega) = 2^{-\omega}$$

$$X_n(\omega) = \begin{cases} 2^n & \text{if } \omega = n \\ 0 & \text{if } \omega \neq n \end{cases}$$ (also denoted $X_n(\omega) = 2^n \delta_{\omega,n}$)

Then $X_n \to 0$ with prob. 1 but

$$E[X_n] = 2^n \cdot 2^{-n} + 0 \cdot 2^{-\omega} = 1 \not\to 0 \text{ as } n \to \infty$$
2 main results give conditions under which it is true that \( E[X_n] \to E[X] \) as \( n \to \infty \)

- **Monotone Convergence Thm**
  (R Thm 4.2.1) \( E[X_n] \to \infty \) if \( \{X_n\} \to \{X\} \) ...

- **Bounded Convergence Thm**
  (R Thm 7.3.1) \( \exists K \in \mathbb{R} \) s.t. \( |X_n| \leq K \) \( \forall n \in \mathbb{N} \) ...

Now we will establish 2 more similar limit thms:

- **Dominated Convergence Thm**
- **Uniformly Integrable Convergence Thm**

First, another result need to prove DCT above.

**Thm 9.1.1 [Fatou's Lemma]**: If \( X_n \geq 0 \) \( \forall n \in \mathbb{N} \) and some constant \( C > -\infty \), then

\[
E\left[ \liminf_{n \to \infty} X_n \right] \leq \liminf_{n \to \infty} E[X_n]
\]

(Allow possibility that both sides are \( \infty \)).

**Pf:** Let \( Y_n = \inf_{k \geq n} X_k \) and \( Y = \lim_{n \to \infty} Y_n = \liminf_{n \to \infty} X_n \).

(\( Y_1 = \inf_{k \geq 1} X_k \), \( Y_2 = \inf_{k \geq 2} X_k \), \ldots)
Then $Y_n \geq C$ (since $X_n \geq C \forall n$ by assumption) and $\exists Y_n \uparrow Y$. Also, $Y_n \leq X_n$ (by defn of $Y$).

By order-preserving property $\Rightarrow$ MCT, it follows that

$$\lim_{n \to \infty} E[Y_n] = E[Y] \quad \text{(by MCT)}$$

$$\liminf_{n \to \infty} E[Y_n] \quad \text{since the limit exists AND } E[X_n] \geq E[Y_n] \quad \text{(order-preserving)}$$

$$\Rightarrow \quad \liminf_{n \to \infty} E[X_n] \geq \liminf_{n \to \infty} E[Y_n] = E[Y] = E\left[\liminf_{n \to \infty} X_n\right] \quad \text{by def of } Y.$$ 

Note: $\liminf_{n \to \infty} X_n$ is interpreted pointwise

i.e. its value at $\omega$ is $\liminf_{n \to \infty} X_n(\omega)$.

* Thm 9.1.2 [Dominated Convergence Thm]: If $X_1, X_2, \ldots$ are RVs, and if $X_n \to X$ with prob. 1 , and if $Y$ s.t. $|X_n| \leq Y \forall n \in \mathbb{N}$ and $E[Y] < \infty$, then

$$\lim_{n \to \infty} E[X_n] = E[X].$$
Pf: First note that \( Y + X_n \geq 0 \). Apply Fatou's Lemma to \( \{Y + X_n\} \), we see that

\[
E[Y] + E[X] = E[Y + X] \leq \liminf_n E[Y + X_n] = E[Y] + \liminf_n E[X_n].
\]

Since \( E[Y] < \infty \), it follows that (cancel \( E[Y] \) terms)

\[
E[X] \leq \liminf_n E[X_n].
\]

Similarly, \( Y - X_n \geq 0 \). Apply Fatou's Lem. to \( \{Y - X_n\} \):

\[
E[Y] - E[X] \leq E[Y] + \liminf_n E[-X_n]
\]

\[
= E[Y] + \limsup_n E[X_n]
\]

\[
\Rightarrow E[X] \geq \limsup_n E[X_n].
\]

However, we always have \( \limsup_n E[X_n] \geq \liminf_n E[X_n] \).

Thus, combining

\[
\limsup_n E[X_n] \leq E[X] \leq \liminf_n E[X_n]
\]

\[
\Rightarrow \limsup_n E[X_n] = \liminf_n E[X_n] = E[X]. \quad \blacksquare
\]

\[
\lim_{n \to \infty} E[X_n]
\]
NOTE: If RV $Y$ is constant, then DCT reduces to BCT.

**Def:** A collection $\{X_n\}$ of random variables is **uniformly integrable** if \[ \lim_{\alpha \to \infty} \sup_n E[|X_n| \mathbf{1}_{|X_n| > \alpha}] = 0. \]

**Note:** Uniform integrability $\Rightarrow$ boundedness of certain expectations

**Prop 9.1.5:** If $\{X_n\}$ is uniformly integrable, then \[ \sup_n E[|X_n|] < \infty. \] Furthermore, if also $X_n \xrightarrow{a.s.} X$ as $n \to \infty$, then $E[|X|] < \infty$.

**Thm 9.1.6 [Uniform Integrability Convergence Thm]:**

If $X_1, X_2, \ldots$ are RVs, and if $X_n \xrightarrow{w} X$ with prob. 1, and if $\{X_n\}$ are uniformly integrable, then \[ \lim_{n \to \infty} E[X_n] = E[X]. \]

**PF (Idea):** Let $Y_n = |X_n - X|$ so that $Y_n \to 0$ as $n \to \infty$. Show that $E[Y_n] \to 0$ as $n \to \infty$ by considering $Y_n$ in 2 pieces: $Y_n = Y_n \mathbf{1}_{Y_n < \alpha} + Y_n \mathbf{1}_{Y_n \geq \alpha}$. 
Then, by triangle inequality
\[ |E[X_n] - E[X]| \leq E[|Y_n|] \to 0 \text{ as } n \to \infty \] which proves the theorem.

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**Moment Generating Functions & Large Deviations**

(Ref: R §9 ch't, B §9)

An interesting connection w/ SLLNs is to estimate the rate at which \( \frac{S_n}{n} \) converges to the mean \( \mu \).

Proof of SLLN used upper bounds for probabilities
\[ P \left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) \text{ for large } \varepsilon. \]

Accurate upper & lower bounds for these probabilities lead to the **Law of the Iterated Logarithm**, a theorem giving precise rates for \( \frac{S_n}{n} \to \mu \).

First, we'd like to estimate the prob. of large deviations from the mean

→ requires use of moment generating functions.