Continuous-time Markov Chains (continued)

Transition Probability Function

Let \( P_{ij}(t) = P(X_{t+s} = j \mid X_s = i) \) denote the probability that a process presently in state \( i \) will be in state \( j \) an amount of time \( t \) later.

\[ \Rightarrow \text{These are the transition probabilities of the CTMC.} \]

(In discrete-time MCs, transition probabilities \( P_{ij} \) is the probability of jumping from \( i \) to \( j \) in 1 step.

In continuous time, there's no first time \( t > 0 \) so we use the above definition of \( P_{ij}(t) \) for each \( t > 0 \).)

We can explicitly determine \( P_{ij}(t) \) in certain cases.

Example: \( \{N(t) : t \geq 0\} \) is a Poisson process with rate \( \lambda \).

Let \( Y_n \) be a discrete-time MC with transition probabilities \( u_{ij} \). Then \( \{X_t = Y_{N(t)}\} \) is a CTMC that takes 1 jump according to \( u_{ij} \) at each arrival of \( \{N(t)\} \).
In this example, $N(t)$ has a Poisson number of jumps with rate $\lambda t$. Hence,

\[ P_{ij}(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} u_{ij}^n \]

where $u_{ij}^n$ is the $n^{th}$ power of the discrete-time MC $Y_n$'s transition prob. $u_{ij}$

Recall: Chapman-Kolmogorov Equation

\[ \sum_k P_{ik}(s) P_{kj}(t) = P_{ij}(s+t) \]

Note: Transition Probabilities $P_{ij}(t)$ can be determined from their derivatives at 0:

\[ q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h)}{h} \quad \text{for } j \neq i \]

If this limit exists (it will for all cases we consider), we call $q_{ij}$ the jump rate from $i$ to $j$. 
In most cases, it is simpler to describe a CTMC by describing its transition rates \( q_{ij} \) for \( i \neq j \) rather than first figuring out \( P_{ij}(t) \).

**Recall**: The rate at which the chain leaves state \( i \) is denoted by \( \nu_i \) s.t.

\[
\nu_i = \sum_{j \neq i} q_{ij}
\]

**Def**: The generator of the CTMC is the matrix \( Q \) such that

\[
Q_{ij} = \begin{cases} 
q_{ij} & \text{if } i \neq j \\
-\nu_i & \text{if } i = j
\end{cases}
\]

aka Instantaneous Transition Rate Matrix

**Def**: State \( i \) is absorbing if \( \nu_i = 0 \).

**Def**: State \( i \) is stable if \( 0 < \nu_i < \infty \).

(If \( \nu_i = \infty \), then the chain will leave \( i \) immediately so we will always assume \( \nu_i < \infty \).)
Kolmogorov's Equations

Theorem: Assume \( \nu_i < \infty \) \( \forall i \) in the state space. Then the transition probabilities are differentiable (in \( t \)), and for any pair of states \( i \neq j \)

\[
\frac{dP_{ij}(t)}{dt} = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t) \quad \text{Backward Equation}
\]

\[
\frac{dP_{ij}(t)}{dt} = \sum_{k \neq j} q_{kj} P_{ik}(t) - \nu_j P_{ij}(t) \quad \text{Forward Equation}
\]

In Matrix notation,

\[
\begin{align*}
P'(t) &= Q P(t) & \text{Backward Eqn} \\
P'(t) &= P(t) Q & \text{Forward Eqn}
\end{align*}
\]

\( \text{Boundary Condition} \quad P(0) = I \)

Rate matrix \( Q \) gives us everything we need to construct a CTMC, if we only had \( P \) we couldn't infer \( Q \).

Example: Backward Eqns for Pure Birth Process \( \lambda_i \) 

\[
P_{ij}'(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t)
\]
Stationary Distribution

Similar to discrete-time MC case, \( \pi \) is said to be the stationary distribution if

\[
\pi P = \pi.
\]

**def:** Let \( \{X_t : t \geq 0\} \) be a CTMC with generator \( Q \).
Then \( \pi \) is a stationary distribution if

\[
\pi Q = 0
\]

OR, equivalently, for each \( j \in S \) the following equations hold:

\[
\sum_{i \neq j} \pi(i) q_{ij} = \nu_i \pi(j)
\]

**def:** We say \( \pi \) satisfies detailed balance for the CTMC if for each \( i \neq j \)

\[
\pi(i) q_{ij} = \pi(j) q_{ji}
\]

**Thm:** If \( \pi \) satisfies detailed balance, then \( \pi \) is a stationary distribution.
Gillespie's Stochastic Simulation Algorithm

Procedure for simulating sample paths of CTMCs.

Reactions: single instantaneous events changing at least 1 of the populations cause the system to change over time.

e.g. birth, death, movement, infection, etc.

Stochastic simulation algorithm (SSA):
samples the time $T$ to the next reaction & updates the system accordingly.

Each reaction $R_j$ is characterized by 2 quantities:

- state change vector $V_j$
  
  $V_j = (v_{i_j}, \ldots, v_{n_j})$

  where $V_{ij}$ is the change in # of indivs in pop. $i$ caused by 1 reaction of type $j$

- propensity function $a_j(x)$

  i.e. prob. that reaction $R_j$ will occur in $[t, t+dt]$ is $a_j(x) \, dt$

  $\frac{rate \ q_j}{\sum all \ rates}$ this is the correct probability