Cor 11.1.7 [Fourier Uniqueness Thm]: Let $X \neq Y$ be RVs.

Then $\phi_X(t) = \phi_Y(t) \forall t \in \mathbb{R}$ iff $L(X) = L(Y)$.

(i.e. iff $X \neq Y$ have the same distribution).

**PF:** ($\implies$) Suppose $\phi_X(t) = \phi_Y(t) \forall t \in \mathbb{R}$. Then (by Inversion Thm)

$$P(a \leq X \leq b) = P(a \leq Y \leq b)$$
provided that

$$P(X = a) = P(X = b) = P(Y = a) = P(Y = b) = 0,$$

i.e. for all but countably many choices for $a \neq b$.

Now take limits and use continuity of probabilities:

$$\implies P(X \in I) = P(Y \in I) \text{ for all intervals } I \subseteq \mathbb{R}.$$

It follows from Prop 2.5.8 (uniqueness of extensions of prob. meas.)

that $L(X) = L(Y)$.

($\impliedby$) Conversely, suppose $L(X) = L(Y)$. Then

$$E[e^{itX}] = E[e^{itY}] \text{ by Cor 6.1.3},$$

$$\implies \phi_X(t) = \phi_Y(t) \forall t \in \mathbb{R}$$

by definition.

$$E[f(X)] = E[f(Y)] \forall \text{ Borel-meas } f: \mathbb{R} \to \mathbb{R} \text{ s.t. exp. value is well-defined}.$$
Need further results in order to prove the continuity theorem.

**Lemma II.1.8 [Helly Selection Principle]:** Let \( \{F_n\} \) be a sequence of CDFs \( F_n(x) = \mu_n((-\infty, x]) \) for some probability distribution \( \mu_n \). Then there is a subsequence \( \{F_{n_k}\} \) and a non-decreasing right-continuous function \( F \) with \( 0 \leq F \leq 1 \) s.t. \( \lim_{k \to \infty} F_{n_k}(x) = F(x) \) \( \forall x \in \mathbb{R} \) s.t. \( F \) is continuous at \( x \).

**[pf: SKIP]**

Note: This lemma does ensure that
\[
\lim_{x \to \infty} F(x) = 1 \quad \text{or} \quad \lim_{x \to -\infty} F(x) = 0 \quad (\text{as we'd expect from a CDF})
\]

To get around this, define a new term: **tight**

**def:** A collection \( \{\mu_n\} \) of probability measures on \( \mathbb{R} \) is \underline{tight} if \( \forall \varepsilon > 0 \), \( \exists a < b \) with \( \mu_n([a,b]) \geq 1 - \varepsilon \) \( \forall n \).

In words, all of the measures have "most" of their mass attributed to the same finite interval \([a,b]\). Mass does not escape off to \( \infty \).
Example: Let \( \mu_n = \delta_{n \mod 3} \) — point mass at \( n \mod 3 \).

i.e. \( \mu_1 = \delta_1 \)
\( \mu_2 = \delta_2 \) \{ repeats \}
\( \mu_3 = \delta_0 \)
\( \mu_4 = \delta_1 \)
\( \mu_5 = \delta_2 \)
\( \mu_6 = \delta_0 \)

Q. Is \( \mu_n \) tight?

Yes. Take \([a,b] = [0,1] \).

Then \( \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \),
\[ \mu_n ([0,1]) = \delta_n ([0,1]) \mod 3 \]
\[ = \frac{1} {2} \mathbb{1}_{[0,1]} (n) \]
\[ = 1 > 1 - \varepsilon. \]

Example: For \( n \geq 1 \), let \( X_n \sim \text{Uniform} (a_n, 2 + a_n) \)

where \( a_n = (-1)^n \). Then

\( X_1 = U(-1, 2 - 1 = 1) \) so \( |X_1| \leq 1 \)
\( X_2 = U(1, 2 + 1 = 3) \) so \( |X_2| \leq 3 \)
\( X_3 = U(-1, 1) \)
\( X_4 = U(1, 3) \)
\[ \Rightarrow |X_n| \leq 3 \ \forall n \in \mathbb{N} \]

\[ \Rightarrow \sup_{n \geq 1} P(|X_n| > 3) < \varepsilon \ \forall \varepsilon > 0 \]

\[ \Rightarrow \text{seq. of prob. distns of } \{X_n\} \text{ is tight} \]
Properties

- Any finite collection of probability measures is tight.
- Union of 2 tight collections of prob. meas. is tight.
- Any sub-collection of a tight collection is tight.

Thm 11.1.10: If \( \{\mu_n\} \) is a tight sequence of probability measures, then there is a subsequence \( \{\mu_{n_k}\} \) of a prob. measure \( \mu \) s.t. \( \mu_n \to \mu \) as \( n \to \infty \).

i.e. \( \{\mu_n\} \) converges weakly to \( \mu \)

PF Idea: By Helly Selection Principle, \( \exists F_{n_k} \preceq F \) s.t.

\[ F_{n_k}(x) \to F(x) \] at all continuity points of \( F \).

Using tightness, can show that \( F \) is actually a prob. distribution function (i.e., in particular \( \lim_{x \to \infty} F(x) = 1 \), \( \lim_{x \to -\infty} F(x) = 0 \), as desired).

Cor 11.1.11: Let \( \{\mu_n\} \) be a tight seq. of prob. dists on \( \mathbb{R} \). Suppose that \( \mu \) is the only possible weak limit of \( \{\mu_n\} \), meaning \( \mu_{n_k} \to \nu \) implies that \( \nu = \mu \).

Then \( \mu_n \to \mu \) as \( n \to \infty \).
One last result: sufficient condition for a sequence of measures to be tight

Lemma 11.1.13: Let \{μₙ\} be a sequence of prob. measures on \(\mathbb{R}\) with characteristic functions \(φₙ(t) = \int e^{itx} \, dμₙ(x)\).

Suppose \(g\) is a function \(g\) (continuous at 0) such that \(\lim_{n \to \infty} φₙ(t) = g(t)\) for each \(|t| < t₀\) for some \(t₀ > 0\). Then \(\{μₙ\}\) is tight.

Theorem 11.1.14 [Continuity Thm]: Let \(μ, μ₁, μ₂, \ldots\) be prob. measures with corresponding characteristic functions \(φ, φ₁, φ₂, \ldots\).

Then \(μₙ \Rightarrow μ\) iff \(φₙ(t) \to φ(t)\) for all \(t \in \mathbb{R}\).

In words: prob. measures converge weakly to \(μ\) iff their char. functions converge pointwise to that of \(μ\).

Pf: First suppose that \(μₙ \Rightarrow μ\) as \(n \to \infty\). Then since \(\cos(tx)\) and \(\sin(tx)\) are bounded continuous functions, we have that

\[
φₙ(t) = \int \cos(tx) \, dμₙ(x) + i \int \sin(tx) \, dμₙ(x)
\]

\[
\to \int \cos(tx) \, dμ(x) + i \int \sin(tx) \, dμ(x)
\]

\[
= φ(x) \quad \text{as } n \to \infty \quad \text{for each } t \in \mathbb{R}
\]
Conversely, suppose that $\phi_n(t) \to \phi(t)$ for each $t \in \mathbb{R}$. Then by using $g = \phi$ in Lemma 11.1.13, the $\mu_{n_k}$ are tight. Now suppose that $\mu_{n_k} \Rightarrow \nu$ for some subsequence $\mu_{n_k}$ of some measure $\nu$. Then

$$\phi_{n_k}(t) \to \phi_{\nu}(t) \quad \forall t \in \mathbb{R}$$

where $\phi_{\nu}(t) = \int e^{itx} d\nu(x)$.

On the other hand, we know that (by assumption)

$$\phi_{n_k}(t) \to \phi(t) \quad \forall t \in \mathbb{R}.$$  

Hence, $\phi_{\nu} = \phi$. By Fourier uniqueness (Cor 11.1.7), this implies that $\nu = \mu$.

Thus, $\mu$ is the only possible weak limit of $\mu_{n_k}$ so by Cor 11.1.11, it follows that $\mu_n \Rightarrow \mu$ as $n \to \infty$. \[\square\]
At def of "tight" measure: \( \forall \varepsilon > 0 \exists M(\varepsilon, x) \ni \sup_{n \geq 1} \mu_n([E, M]) < \varepsilon \)

**Ex 1.** Let \( \mu_n \) be the \( \delta \) at \( n \mod 3 \) so it rotates through \( S_1, S_2, S_3 \), etc. Is it tight? Yes — we can define a finite interval \( t \) over which all the mass on it takes \( \gamma \), and \( \forall \varepsilon, \mu_n([0, 2]) = 1 \geq 1 - \varepsilon \)

**Ex 2.** For \( n \in \mathbb{N} \), let \( X_n \sim \text{Unif}(a_n, 2 + a_n) \) where \( a_n = (-1)^n \)

**Lemma 1.1.13**

For \( x \in \mathbb{R} \), let \( \mu_n \) be \( \text{Unif}(\frac{(-1, 1)}{2}) \) for \( n \) odd and \( \text{Unif}(1, 3) \) for \( n \) even. 

```
\[ \sup \mu_n(\mathbb{R}) = 0 < \varepsilon \]
```

One last result: sufficient condition for \( \{ \mu_n \} \) to be tight: \( \text{Lemma 11.1.14} \)

For \( \mu \) an \( \mu \)-finite, \( \mu \)-continuous \( f \) \( \text{vanishing at } 0 \), \( \forall \varepsilon \), we have

(3) \( \text{such that } \lim_{n \to \infty} \mu_n(\mathbb{R}) = g(\mathbb{R}) \), then \( \mu_n \) is tight.

**Theorem 11.1.14**

Continuity Theorem:

Let \( \mu_1, \mu_2, \mu_3, \ldots \) be the product of measures \( \{ f, f_1, f_2, \ldots \} \)

then \( \mu_n \Rightarrow \mu \) iff \( \mu_n(t) \to \mu(t) \)

Weaker convergence (pointwise)

(i.e. in law)

**Proof:** \( \mu \to \mu \); then since \( cos(tx), sin(tx) \) are bounded & continuous, we have

\[ \mu_n(\mathbb{R}) = \int e^{itx} \, d\mu_n(x) \to \int e^{itx} \, d\mu(x) \]

(by Fourier Uniqueness Theorem)

Conversely, \( \mu_n(t) \to \mu(t) \) (pointwise); then using \( g = g \) in Lemma 11.1.13, we know the \( \mu_n \) are tight. Now suppose \( \mu_{n_k} \Rightarrow \mu \) for some measure \( \mu \), then

\[ \mu_{n_k}(t) \to \mu(t) \quad \forall t \in \mathbb{R} \]

We know \( \mu_{n_k}(t) \to \mu(t) \), hence \( \mu = \mu \)

By inverse uniqueness \( \mu_{n_k} \), we get \( \mu = \mu \), and \( \mu \) is unique (weak) limit of \( \mu_n \); so by (11.1.11), \( \mu_n \Rightarrow \mu \)