Renewal Theory

[Ref: Models Ch. 7]

Recall: A Poisson process is a counting process for which the times between successive events are i.i.d. exponential RVs.

- One generalization is to consider a counting process for which the times between successive events are i.i.d. with an arbitrary distribution.

- Such a counting process is called a renewal process.

**def:** Let \{N(t) : t \geq 0\} be a counting process if let \(X_n\) be the time between the \((n-1)\)st and \(n\)th event, \(n \geq 1\). If the sequence \(\{X_1, X_2, \ldots\}\) is i.i.d., then the counting process \{N(t) : t \geq 0\} is called a renewal process.

* When an event occurs, we say a renewal has taken place. Process starts itself over.
Example: Replacement of lightbulbs

Suppose we have an infinite supply of lightbulbs whose lifetimes are i.i.d.

We use a single lightbulb at a time, and when it fails, we immediately replace it with a new one.

\( \{N(t) : t \geq 0\} \) is a renewal process when

\[ N(t) = \# \text{ of lightbulbs that have failed by time } t. \]

Event = failure of lightbulb

Renewal Process \( \rightarrow \) replace bulb immediately upon failure

Inter-event Time = lifetime of a bulb

* Each event starts the process over with identical condition (aka "renewal")

\[ S_0 = 0 \]
\[ S_n = \sum_{i=1}^{n} X_i, \ n \geq 1 \]

/ \ waiting time until the \( n^{th} \) event (renewal)

\[ X_1 \]
\[ X_2 \]
\[ X_3 \]

0 \( \rightarrow \)

\( S_1 \)
\( S_2 \)
\( S_3 \)

time

time of 1st renewal = \( X_1 \)
time of 2nd renewal = \( X_1 + X_2 \)
Let $F$ denote the interarrival distribution ($X_i \sim F$).

Let $\mu = E[X_n]$ - mean time b/t successive renewals

- By the strong law of large numbers,
  \[
  \frac{S_n}{n} \to \mu \quad \text{as} \quad n \to \infty
  \] (where $S_n$ is the time of $n^{th}$ renewal)
  with probability 1

- $N(t) = \max \left\{ n : S_n \leq t \right\}$

- $N(t)$ is finite for each $t$ (with prob. 1)

- $N(\infty) = \text{total \# of renewals} = \infty$ (with prob. 1)

  \[
  \left( \lim_{n \to \infty} N(t) \right)
  \]

Relationship Between $N(t)$ and $S_n$

\[
N(t) \geq n \iff S_n \leq t
\]

\[
\begin{array}{c|c|c}
\text{\# of renewals} & \text{time of } n^{th} \text{ renewal} \\
\text{by time } t \geq n & \leq t
\end{array}
\]
\[ P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n+1) \]

\[ = P(S_n \leq t) - P(S_{n+1} \leq t) \]

allows us to compute the distribution of \( N(t) \) from the distn of \( X_i \)'s

\[ \text{Mean Value of } N(t) \]

\[ E[N(t)] = \sum_{n=1}^{\infty} n \, P(N(t) = n) = \sum_{n=1}^{\infty} P(N(t) \geq n) \]

\[ = \sum_{n=1}^{\infty} P(S_n \leq t), \]

\[ \uparrow \quad \text{definition} \]

\[ \uparrow \quad \text{Alternative formula for integer-valued RVs} \ (\geq 0) \]

\[ \text{Def: The mean-value function or renewal function of a renewal process is} \]

\[ m(t) = \sum_{n=1}^{\infty} P(S_n \leq t) \]

Notes:
1. \( m(t) \) uniquely determines the renewal process, \( \{N(t)\} \not\perp X_i \)'s.
2. \( \{N(t)\} \) as a Poisson process

\[ \Rightarrow m(t) = \lambda t = \frac{t}{\mu} \text{ where } \mu = \frac{1}{\lambda} = E[X_i] \]
Thus, (by (1) above) the Poisson process is the only renewal process having a linear mean-value function.

\[ m(t) = \lambda t \quad \text{linear function of } t \]

**Renewal Equation**

(continuous)

\[ X_i \text{ - interarrival times, PDF } f \text{ and CDF } F \]

\( \{ N(t) : t \geq 0 \} \) - corresponding renewal process with renewal function \( m(t) = E[N(t)] \)

\[ m(t) = F(t) + \int_0^t m(t-x) f(x) \, dx \]

This is the renewal equation

(often used to obtain \( m(t) \))

**Details:** Condition on time of 1st renewal

\[ m(t) = E[N(t)] = \int_0^\infty E[N(t) \mid X_1 = x] f(x) \, dx \]

2 cases:

\[ \begin{cases} x \leq t & \Rightarrow N(t) > 1 \\ x > t & \Rightarrow N(t) = 0 \end{cases} \]
Then, \[ E[N(t) \mid X_1 = x] = 1 + E[N(t-x)] \] if \( x \leq t \)
\[ E[N(t) \mid X_1 = x] = 0 \] if \( x > t \)

\[ m(t) = \int_0^t E[N(t) \mid X_1 = x] f(x) \, dx \]

\[ = \int_0^t \left[ 1 + E[N(t-x)] \right] f(x) \, dx \]
\[ = \int_0^t f(x) \, dx + \int_0^t m(t-x) f(x) \, dx \]
\[ = F(t) + \int_0^t m(t-x) f(x) \, dx \]
\[ \text{by def} \]

Example: \( X_i \sim \text{Uniform}(0,1) \)

One instance in which the renewal eqn may be solved: interarrival distn \( \sim U(0,1) \)

\[ m(t) = e^t - 1 \] for \( 0 \leq t \leq 1 \)

(see book for details!)
Limit Theorems

We saw that, with probability 1,\[ N(t) \to \infty \text{ as } t \to \infty \]

Q. How fast does this happen? 
(At what rate does \( N(t) \) go to \( \infty \)?) 
\[ \lim_{t \to \infty} \frac{N(t)}{t} \ ? \]

- For a Poisson process \( \{N(t) : t \geq 0\} \),
  \[ E[N(t)] = m(t) = \lambda t = \frac{t}{\mu} \text{ where } \mu = \frac{1}{\lambda} = E[X_1] \]
  \( X_i \)'s \sim \exp(\lambda) \]

\[ \frac{N(t)}{t} \to \lambda = \frac{1}{\mu} \text{ a.s. as } t \to \infty \]

\[ \left( \text{Note: } E\left[ \frac{N(t)}{t} \right] = \frac{\lambda t}{t} = \lambda \right) \]

- This is true in general for a renewal process!

\[ \frac{N(t)}{t} \overset{\text{a.s.}}{\to} \frac{1}{\mu} \text{ as } t \to \infty \]

where \( \mu = E[X_1] \)
and \( X_i \)'s \ i.i.d.
Proof: See details in §7.3 (Models Book)