Note: There also exist uncountable sets with Lebesgue measure 0.

Simplest Example: Cantor Set $K$

Begin with $[0,1]$. Then remove the open interval $(\frac{1}{3}, \frac{2}{3})$.

Continue removing the open middle intervals of these 2 pieces, etc.

The complement of the Cantor set $K^c$ has Leb. meas. $= 1$

$$\lambda(K^c) = \frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1.$$
Thus, since \( P(A) = 1 - P(A^c) \), we have

\[
P(K) = 1 - P(K^c) = 1 - 1 = 0.
\]

\( K \) is uncountable (see justification in Rosenthal).

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Extensions of the Extension Thm

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Uniqueness Property:

Prop: Let \( J, P, P^* \) and \( M \) be as in the extension theorem. Let \( \mathcal{F} \) be any \( \sigma \)-algebra with \( J \subseteq \mathcal{F} \subseteq M \) (e.g. \( \mathcal{F} = M \) or \( \mathcal{F} = \sigma(J) \)).

Let \( Q \) be any probability measure on \( \mathcal{F} \) s.t. \( Q(A) = P(A) \) \( \forall A \in J \).

Then \( Q(A) = P^*(A) \) \( \forall A \in \mathcal{F} \).

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Useful special case: \( \mathcal{F} = \mathcal{B} \) - Borel subsets of \( \mathbb{R} \) (or of \([0,1]\))
Random Variables

(ref §5, Rosenthal §3) Although Billingsley focuses on simple RVs, those with finite range.

Main Idea: If we think of \( \Omega \) as the set of all possible random outcomes of an experiment, then a random variable assigns a numerical value to these outcomes.

def: Given a probability triple (prob. space) \((\Omega, \mathcal{F}, P)\), a simple random variable is a function \( X \) from \( \Omega \) to \( \mathbb{R} \) such that

\[ \{ \omega \in \Omega : X(\omega) = x \} \in \mathcal{F}, \quad x \in \mathbb{R}. \]

In other words, the function \( X \) must be measurable.

Could also write:

\[ \{ X = x \} \in \mathcal{F}, \quad \forall x \in \mathbb{R} \]

\[ X^{-1}(\{ x \}) \in \mathcal{F}, \quad \forall x \in \mathbb{R} \]

For general RV (not necessarily simple):

\[ \left\{ X \in B \right\} \in \mathcal{F}, \text{ for every Borel set } B \]

\[ X^{-1}(B) \in \mathcal{F}, \text{ for every Borel set } B \]
Note: Not all functions from $\Omega$ to $\mathbb{R}$ are RVs.

Example: Let $(\Omega, \mathcal{F}, P)$ be Lebesgue measure on $[0,1]$, and let $H \subseteq \Omega$ be the non-measurable set from Lecture 1 (uses equiv. classes & shift operator). Define $X: \Omega \to \mathbb{R}$ by $X = 1_H$, so
\[
\begin{cases}
X(\omega) = 0 & \text{for } \omega \in H \\
X(\omega) = 1 & \text{for } \omega \not\in H.
\end{cases}
\]

Then $\{\omega \in \Omega : X(\omega) = \frac{1}{2}\} = H$ but $H \not\in \mathcal{F}$, so $X$ is not a RV.

Proposition: Given $(\Omega, \mathcal{F}, P)$.

(i) If $X = 1_A$ is the indicator function of some event $A \in \mathcal{F}$, then $X$ is a random variable.

(ii) If $X \leq Y$ are RVs and $c \in \mathbb{R}$, then
\[
\{X + c, X + Y, cX, XY, X^2\}
\]
are all RVs.

(iii) If $Z_1, Z_2, \ldots$ are RVs s.t. $\lim_{n \to \infty} Z_n(\omega)$ exists for each $\omega \in \Omega$ and $Z(\omega) = \lim_{n \to \infty} Z_n(\omega)$, then $Z$ is also a RV.
More details on Random Variables

**Def:** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real-valued function $X$ on $\Omega$ is a random variable if

$$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \quad \text{for each } B \in \mathcal{B},$$

short hand is

$$\{x \in B\} \quad \text{or } X^{-1}(B) \in \mathcal{F} \quad \text{Borel sets}$$

**Note:** We are interested in the probability that $X$ is a member of $B$ for each Borel set $B$, i.e.

$$P(\{\omega \in \Omega : X(\omega) \in B\}) = P(\{x \in \mathbb{R} \mid X(x) \in B\}) = P(x \in B)$$

But for this probability to exist, $\{x \in \mathbb{R} \mid X(x) \in B\}$ must be an event.

Hence the defn of RV above.

**Remark:** Recall that a real-valued function $f$ on $\Omega$ is $\mathcal{F}$-measurable iff $f^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{B}$.

Thus, random variables are just $\mathcal{F}$-measurable functions.
Measurability of a function depends only on the \( \sigma \)-algebra \( \mathfrak{F} \) and not on the prob. measure \( P \).

One of the most important quantities associated with a RV is its **probability distribution**.

**def:** Let \( X \) be a RV on the probability space \((\Omega, \mathfrak{F}, P)\). Then the **probability distribution** of \( X \), denoted \( \mu_X \), is the set function on \( \mathcal{B} \) defined by

\[
\mu_X(B) = P(X \in B), \text{ for } B \in \mathcal{B}.
\]

* e.g. PMF for discrete RV
* e.g. PDF for continuous RV

**def:** **Probability distribution function** (aka CDF) of \( X \):

\[
F_X(x) = P(X \leq x), \quad \forall x \in \mathbb{R}
\]
Pf: (i) If $X = 1_A$, for $A \in \mathcal{A}$, then for any subset $B \in \mathcal{A}$, $X^{-1}(B)$ must be one of the following: $A$, $A^c$, $\emptyset$, or $\Omega$. Hence $X^{-1}(B) \in \mathcal{A}$.

(See Rosenthal for remaining proofs!)

Suppose now that $X$ is a RV and $f: \mathbb{R} \to \mathbb{R}$ is a function which is Borel-measurable, meaning that $f^{-1}(A) \in \mathcal{B}$ for any $A \in \mathcal{B}$ where $\mathcal{B} = \text{collection of Borel sets of } \mathbb{R}$.

[Equivalently, $f$ is a RV corresponding to $\Omega = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}$]

Define a new RV $f(X)$ by

\[
\begin{align*}
  f(X)(\omega) &= f(X(\omega)) \\
  \quad \uparrow \\
  \text{the composition} \\
  \text{of } X \text{ with } f
\end{align*}
\]

for each $\omega \in \Omega$.

Note: this is well-defined since for $B \in \mathcal{B}$,

\[
\{f(X) \in B\} = \{X \in f^{-1}(B)\} \in \mathcal{A}
\]
Prop: If $f$ is a continuous function, or a piecewise-continuous function, then $f$ is Borel-measurable.

Pf: A basic result of point-set topology says:
if $f$ is continuous, then $f^{-1}(O)$ is an open subset of $\mathbb{R}$ whenever $O$ is. In particular,
$f^{-1}((x,\infty))$ is open, so $f^{-1}((x,\infty)) \in B$, so $f^{-1}([\infty,x)) \in B$.

If $f$ is piecewise-contin. / then we can write

$$f = f_1 1_{I_1} + f_2 1_{I_2} + \ldots + f_n 1_{I_n}$$

where $f_j$'s are continuous and the $\{I_j\}$ are disjoint intervals. It follows from above & prev. prop. (3.1.5) that $f$ is $B$-measurable.

Example: $f(x) = x^k$ for $k \in \mathbb{N}$

$f$ is Borel-measurable.
Thus if $X$ is a RV, then so is $X^k \forall k \in \mathbb{N}$. 