Recall defn of Random Variable: (Ref: R Analy. McDonald Weiss)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real-valued function $X$ on $\Omega$ is a random variable if

$$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \text{ for each Borel set } B \in \mathcal{B}.$$ 

Examples:

1. $X$ is a discrete RV if $\exists$ a countable set $K$ s.t. $P(X \in K) = 1$. We can write $K = \{x_n : n \in \mathbb{N}\}$. Then the probability mass function (PMF) of $X$ is $P_X : \mathbb{R} \to [0,1]$ defined by $p_X(x_n) = P(X = x_n)$. Note that $p_X$ is 0 on $K^c$. Also, $\mu_X(A) = \sum_{x_n \in A} P_X(x_n)$ for $A \subseteq K$.

2. Suppose 2 dice are rolled. Probability space:

$$\Omega = \{(i, j) : i, j = 1, 2, \ldots, 6\}$$

$$\mathcal{F} = P(\Omega) = 2^\Omega \text{ (Power set)}$$

$$P = \frac{8}{36} \text{ where } \delta \text{ is counting measure}$$
Let \( X = \text{sum of the 2 dice.} \)

\( X \) is a discrete RV since \( \mathbb{P}(X \in \{2, 3, \ldots, 12\}) = 1 \) (countable (finite in this case) set \( K \)).

PMF of \( X \) is

\[
\begin{align*}
p_x(x) &= \begin{cases} 
(x-1)/36, & x=2, 3, \ldots, 7 \\
(13-x)/36, & x=8, 9, \ldots, 12 \\
0, & \text{o.w.}
\end{cases}
\end{align*}
\]

3. \( X \) is an absolutely continuous RV if there is a non-negative Borel measurable function \( f \) s.t.

\[
\mu_x(B) = \int_B f \, d\lambda \quad \text{for all } B \in \mathcal{B}.
\]

Typically write \( f = f_x \) and call \( f_x \) the probability density function (PDF) of \( X \). Also, \( \lambda = \text{Lebesgue measure}. \)

4. Suppose a random number is chosen from \([0, 1]\), let \( X \) denote the random # obtained. Then for \( B \in \mathcal{B} \), we have

\[
\mu_x(B) = \mathbb{P}(X \in B) = \lambda(B \cap [0, 1]) = \int_B \chi_{[0,1]} \, d\lambda.
\]

\( X \) is an abs. continuous RV with PDF \( f_x = \chi_{[0,1]} \).
In this case, the characteristic function is the indicator function on $[0, 1]$:

$$X_{[0, 1]} = 1_{[0, 1]} = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{if } x \notin [0, 1] \end{cases}$$

Such a RV has the uniform distribution on $[0, 1]$.

5. $X$ is a continuous RV if

$$P(X = x) = 0 \quad \forall x \in \mathbb{R}.$$  

Note: $P(X \in K) = 0$ for each countable subset $K \subset \mathbb{R}$

Note: An absolutely continuous RV is a continuous RV, but converse is NOT true.

Independence

* Informally, events or random variables are independent if they don't affect each other's probabilities.

**Def:** Two events $E$ & $F$ are independent if

$$P(E \cap F) = P(E) \cdot P(F)$$

If $E \neq F$ are not independent, they are dependent.
For more than 2 events, need to carefully distinguish between pairwise independence and mutual independence.

**Def:** Let \((\Omega, \mathcal{F}, P)\) be a probability space. Events \(A_1, A_2, \ldots, A_n\) are said to be mutually independent if for each subset \(\{i_1, i_2, \ldots, i_m\}\) of \(\{1, 2, \ldots, n\}\), we have

\[
P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_m}).
\]

**Note:** Mutually independent events are pairwise independent but not vice versa.

**Def:** Events \(A_1, \ldots, A_n\) are pairwise independent if for all \(i \neq j\), \(A_i\) and \(A_j\) are independent.

More generally, the events of an arbitrary (possibly infinite) collection \(\{A_i\}_{i \in I}\) are called mutually independent if every finite number of them are mutually indep.
Independent RVs

Intuitively, (consider 2 RVs) 2 RVs are indep. if knowing the value of 1 of the variables does not affect the probability distribution of the other RV.

**Def:** Random variables \( X, Y \) (both defined on \((\Omega, \mathcal{F}, P)\)) are independent if, for all Borel sets \( A, B \), the events \( \{X \in A\} \) and \( \{Y \in B\} \) are independent. That is,

\[
P(X \in A, Y \in B) = P(X \in A) P(Y \in B)
\]

More generally,

*RVs* \( X_1, \ldots, X_n \) defined on the same prob. space \((\Omega, \mathcal{F}, P)\) are mutually independent if

\[
P(X_1 \in B_1, \ldots, X_n \in B_n) = P(X_1 \in B_1) \cdots P(X_n \in B_n)
\]

for all Borel sets \( B_1, \ldots, B_n \).

(As for events, the RVs of an infinite collection are mut. indep. if the RVs of each finite subcollection are mut. indep.)
Can also define **pairwise independence** for RVs:

RVs \( \{X_i\}_{i \in I} \) are pairwise indep. if for each (all defined on same prob. space)

\[ i, j \in I, \quad X_i \text{ and } X_j \text{ are independent.} \]

Again, mutually indep RVs \( \Rightarrow \) pairwise indep RVs but converse is **NOT** true.

**Prop.** Let \( X \perp Y \) be independent RVs. Let \( f, g : \mathbb{R} \to \mathbb{R} \) be Borel-measurable functions. Then the RVs \( f(X) \) and \( g(Y) \) are independent.

**PF:** For Borel sets \( B_1, B_2 \subseteq \mathbb{R} \),

\[
P\left( f(X) \in B_1, g(Y) \in B_2 \right) = P\left( X \in f^{-1}(B_1), Y \in g^{-1}(B_2) \right) 
\]

\[= P\left( X \in f^{-1}(B_1) \right) P\left( Y \in g^{-1}(B_2) \right) \quad \text{(since } f \text{ and } g \text{ are B-maps!)}
\]

\[= P\left( f(X) \in B_1 \right) P\left( g(Y) \in B_2 \right) . \]
Prop: Let \( X \perp Y \) be RVs defined on same prob. space \((\Omega, \mathcal{F}, P)\). Then \( X \perp Y \) are indep. iff
\[
P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y)
\]
i.e. \( F_{X,Y}(x,y) = F_X(x) F_Y(y) \)
(Joint CDF is product of marginal CDFs)

Pf: \( X \perp Y \) indep \( \Rightarrow \) \( \otimes \) by definition.

Now suppose \( \otimes \) holds. Then ... see details in McDonald or Rosenthal.

Continuity of Probabilities
(Ref: § 3.3 Rosenthal, § 4 Billingsley)

Given a probability space \((\Omega, \mathcal{F}, P)\) and events \( A, A_1, A_2, \ldots \in \mathcal{F} \),
\[
\{A_n\} \not\rightarrow A \text{ means that } A_1 \subseteq A_2 \subseteq \ldots \text{ and } \bigcup_n A_n = A.
\]

The events \( A_n \) increase to \( A \).
Similarly, $\{A_n\} \downarrow A$ means that $\{A_n^c\} \uparrow A^c$ or equivalently

$A_1 \supseteq A_2 \supseteq \ldots \text{ and } \bigcap_n A_n = A$.

The events $A_n$ decrease to $A$.

Prop (Continuity of Probabilities): If $\{A_n\} \uparrow A$ or
If $\{A_n\} \downarrow A$, then \[
\lim_{n \to \infty} P(A_n) = P(A).
\]

PF: First suppose that $\{A_n\} \uparrow A$. Let $B_n = A_n \cap A_{n-1}^c$.
Then the $\{B_n\}$ are disjoint with

$\bigcup_n B_n = \bigcup_n A_n = A$.

Hence,

$P(A) = P\left(\bigcup_{m=1}^{\infty} B_m\right) = \bigcup_{m=1}^{\infty} P(B_m)$

\text{ disjoint union}

$= \lim_{n \to \infty} \sum_{m=1}^{n} P(B_m) = \lim_{n \to \infty} P\left(\bigcup_{m \leq n} B_m\right)$

\text{ again this is a disjoint union}

$= \lim_{n \to \infty} P(A_n)$ since $\{A_n\}$

is a nested sequence.

Similarly, suppose that $\{A_n\} \downarrow A$. PF left as an exercise.