Markov Chains
[ref: Models ch. 4]

Discrete-Time Markov Chain vs. Continuous-Time Markov Chain

\[ T = \{0, 1, 2, \ldots\} \]

(or finite subset)

State space \( = \mathbb{N} \) for example

\[
T = [0, \infty)
\]

state space \( = \mathbb{N} \)

\( \star \) Markov chain is a special type of stochastic process which has the Markov Property:

the conditional distribution of the future state given the current state and the past is independent of the past.

In other words,

the probability of any particular behavior of the process, when the current state is known, is not affected by additional knowledge of its past behavior.

e.g. Random Walk (1-dim) on \( \mathbb{Z} \)

\[ X_n = 1 \]

\[ X_{n+1} = 0 \text{ or } 2 \]
doesn't care about \( X_i \) for \( i = 0, 1, \ldots, n-1 \)

\[ \ldots -2 -1 0 1 2 \ldots \]
\[ \{X_n : n = 0, 1, \ldots \} \text{ - discrete-time MC} \]

**Transition Probability** from state \( i \) to state \( j \) at time \( n \) (1-step transition)

\[
P( X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0 )
= P( X_{n+1} = j \mid X_n = i ) \quad \text{(past history)}
= P_{ij}(n) \quad \text{(future state)}
= P_{ij}(n) \quad \text{(current state)}
\]

*Independent of past history!*

\[
P(X_{t+s} = j \mid X_s = i, X_u = x(u), 0 \leq u < s)
= P( X_{t+s} = j \mid X_s = i ) \quad \text{where} \quad i, j, x(u) \geq 0
= P_{ij}(t)
\]

In many examples, the transition probability does not depend on the time \( n \):

\[
\text{If} \quad P( X_{n+1} = j \mid X_n = i ) = P_{ij} = P( X_1 = j \mid X_0 = i ) \quad \text{No dependence on} \ n \ !
\]

then the Markov chain has stationary transition probabilities.
Arrange the $P_{ij}$'s into a **Transition Probability Matrix** $P$

i.e. matrix of 1-step transition probabilities

$$P = \begin{pmatrix}
P_{00} & P_{01} & P_{02} & \cdots \\
P_{10} & P_{11} & P_{12} & \cdots \\
P_{20} & P_{21} & P_{22} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$P_{ij} = P(X_{n+1} = j \mid X_n = i)$

= prob. that the process makes the transition $i \to j$

$P_{ij} \geq 0$ & $\sum_{j=0}^{\infty} P_{ij} = 1$  

for $i=0,1,2,\ldots$ since the process must make a transition into some state each time step

**n-step transition probabilities**

$$P_{ij}^n = P(X_{n+k} = j \mid X_k = i), \ n \geq 0, \ i,j \geq 0$$

\[\text{process moves from state } i \text{ to } j\]

\[\text{in } n \text{ steps (transitions)}\]

Q. Long term behavior of the MC?
\[ P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \]

\[ \Rightarrow P^{(n+m)} = P^n \cdot P^m \]

(Notation \( P^{(k)} = P^k \)

\( k \)-step transition matrix)

: details in book

\[ P^n = P \cdot P \cdots P \] \( n \) times

\[ P^n = \begin{bmatrix}
    p_{00}^n & p_{01}^n & \cdots & p_{0j}^n & \cdots \\
    p_{10}^n & p_{11}^n & \cdots & p_{1j}^n & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    p_{ij}^n & p_{jj}^n & \cdots & p_{jj}^n & \cdots
\end{bmatrix} \]

\( j^{th} \) column: conditional probabilities that the process will be in state \( j \) after \( n \) steps
Example 1: 2-state Markov Chain

\[ P = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \]

move to next state in 1 step

rows sum to 1

current state

Example 2: Wright-Fisher model (genetics)

Model of random genetic drift
change in allele frequency of a gene over time

Consider a constant size population of 2N individuals:
2 types \( A \) allele
\( a \) allele

If the current generation consists of
\[ i \text{ type } A \text{ individuals} \]
\[ 2N-i \text{ type } a \text{ individuals} \]

then the next generation is determined by 2N Bernoulli trials with "success" probability

\[ P_i = \frac{i}{2N} \quad (\text{for an } A \text{ type indiv.}) \]

and \[ 1-P_i = 1-\frac{i}{2N} = \frac{2N-i}{2N} \quad (\text{for an } a \text{ type indiv.}) \]
Define a MC \( \{ X_n : n = 0, 1, \ldots \} \) such that

\[
X_n = \# \text{ of type A alleles in the } n^{th} \text{ generation}
\]

\( S = \{ 0, 1, \ldots, 2N \} \)

\[
\text{i.e. Binomial RV w/parameters } n \& p_i
\]

Transition Probability:

\[
P(X_{n+1} = j \mid X_n = i) = \binom{2N}{j} p_i^j (1-p_i)^{2N-j} \quad \text{for } i, j = 0, 1, \ldots, 2N
\]

Binomial PDF

Note: 0 \& 2N are called absorbing states

i.e. once the process enters 1 of these states (where all individuals are the same type), the process cannot leave that state.

If we modify the model to include mutation:

\[
\mu_1 = P(A \rightarrow a) \quad \frac{1}{2} \mu_2 = P(a \rightarrow A)
\]

then

\[
P_i = \frac{i}{2N} (1-\mu_1) + \left(1-\frac{i}{2N}\right) \mu_2
\]

\( \Rightarrow 0 \& 2N \) are no longer absorbing states

\( \frac{1}{2} \) long-term behavior of MC stabilizes as \( n \rightarrow \infty \).
Example 1

Example 2

Example 3

- modified version of Example 2:

2 is an absorbing state

\[
P = \begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
0 & 1 & 0
\end{bmatrix}
\]

Check:
* Rows of P sum to 1