Tail Fields

(ref: §3.5 Rosenthal, §4 Billingsley)

\[ T = \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \ldots) \]

In words, an event \( A \in T \) must have the property that for any \( n \in \mathbb{N} \), it depends only on the events \( A_n, A_{n+1}, \ldots \). In particular, \( A \in T \) iff changing/omitting a finite number of values does not affect the occurrence of \( A \). (does not depend on any finite set)

\[ \{A_1, A_2, \ldots, A_n\} \]

Examples

- \( \limsup_n A_n \in T \)
- \( \liminf_n A_n \in T \)

Kolmogorov's 0-1 Law: If events \( A_1, A_2, \ldots \) are independent with tail field \( T \), then for \( A \in T \), \( P(A) = 0 \) or \( 1 \).
Expectation

(ref: § 5, Rosenthal § 4)

Simple Random Variables

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $X$ be a random variable defined on this space.

**def:** A random variable $X$ is **simple** if $\text{range}(X)$ is finite, where $\text{range}(X) = \{ X(\omega) : \omega \in \Omega \}$.

It assumes only finitely many values.

We can list the distinct elements in the range of $X$ as $x_1, x_2, \ldots, x_n$

and write $X = \sum_{i=1}^{n} x_i 1_{A_i}$

where $A_i = \{ \omega \in \Omega : X(\omega) = x_i \} = X^{-1}(\{x_i\}) \in \mathcal{A}$

and where $1_{A_i}$ are indicator functions.

Note that the $A_i$'s form a finite partition of $\Omega$. 
For a simple RV $X = \sum_{i=1}^{n} x_i 1_{A_i}$, we define its expected value (or expectation or mean) by

$$E[X] = E\left[\sum_{i=1}^{n} x_i 1_{A_i}\right] = \sum_{i=1}^{n} x_i P(A_i)$$

where $\{A_i\}$ is a finite partition of $\Omega$.

**Notation:** Often write $\mu_X = E[X]$.

**Example:** Let $(\Omega, \mathcal{F}, P)$ be Lebesgue measure on $[0,1]$, and define simple RVs $X$ and $Y$ by

$$X(\omega) = \begin{cases} 5, & \text{if } \omega > \frac{1}{3} \\ 3, & \text{if } \omega \leq \frac{1}{3} \end{cases}$$

$$Y(\omega) = \begin{cases} 2, & \text{if } \omega \text{ is rational } (\not\in [0,1]) \\ 4, & \text{if } \omega = \sqrt{2} (\approx 1.414) \\ 6, & \text{if other } \omega \leq \frac{1}{4} \text{ i.e. } \left[0,\frac{1}{4}\right) \cup \left(\frac{1}{2},\frac{3}{4}\right) \\ 8, & \text{otherwise} \end{cases}$$

Compute $E[X]$ and $E[Y]$. 
Partition of $[0,1]$: $[0, \frac{1}{3}), \frac{1}{3}, 1$

$A_1 \quad A_2$

$$E[X] = 3 \cdot P\left([0, \frac{1}{3})\right) + 5 \cdot P\left(\frac{1}{3}, 1\right)$$

$$= 3 \cdot \frac{1}{3} + 5 \cdot \frac{2}{3} = \frac{3}{3} + \frac{10}{3} = \frac{13}{3}$$

Partition of $[0,1]$: $\emptyset \cap [0,1], \left\{ \frac{1}{4}, \frac{3}{4}\right\}, [0, \frac{1}{4}] \setminus \emptyset \cap [0,1], A_4$

$A_1 \quad A_2 \quad A_3$

where $A_4 = [0,1] \setminus (A_1 \cup A_2 \cup A_3) = \left(\frac{1}{4}, 1\right] \setminus \left(\emptyset \cap [0,1] \cup \left\{ \frac{1}{4}, \frac{3}{4}\right\}\right)$

$$E[Y] = 2 \cdot P(A_1) + 4 \cdot P(A_2) + 6 \cdot P(A_3) + 8 \cdot P(A_4)$$

$\text{countable} \quad \text{singleton} \quad \frac{1}{4} \quad \frac{3}{4}$

$$= \frac{6}{4} + \frac{24}{4} = \frac{30}{4} = \frac{15}{2}$$

Remark: $E[\mathbf 1_A] = P(A)$ for all $A \in \mathcal A$

Simple RV

$E[c] = c \quad \forall c \in \mathbb R$ (constants)

Simple RV

$X(w) = c$
Expectation is Linear

Let \( X = \sum_{i=1}^{n} x_i 1_{A_i} \) and \( Y = \sum_{j=1}^{m} y_j 1_{B_j} \), where 
\[ \{A_i\} \text{ and } \{B_j\} \text{ are finite partitions of } \Omega. \]

If \( a, b \in \mathbb{R} \), then \( \{A_i \cap B_j\} \) is again a finite partition of \( \Omega \) and we have

\[
E[aX + bY] = E\left[ \sum_{i,j} (ax_i + by_j) 1_{(A_i \cap B_j)} \right]
\]

\[
= \sum_{i,j} (ax_i + by_j) P(A_i \cap B_j)
\]

\[
= a \sum_{i} x_i P(A_i) + b \sum_{j} y_j P(B_j)
\]

\[
= a \ E[X] + b \ E[Y].
\]

Note: It follows that \( E\left[ \sum_{i=1}^{n} x_i 1_{A_i} \right] = \sum_{i=1}^{n} x_i P(A_i) \)

for any finite collection of subsets \( A_i \subseteq \Omega \), even if they do not form a partition.

\[
\text{[since } P(A) = \sum_{i} P(A \cap B_i) \text{ for any partition } \{B_i\}.]\]
Expectation is Order Preserving

If simple RVs $X \leq Y$ are s.t. $X \leq Y$ (meaning that $X(w) \leq Y(w)$ $\forall w \in \Omega$), then

$$E[X] \leq E[Y].$$

Why is this true?

$$X \leq Y \implies Y - X \geq 0$$

$$\implies E[Y - X] \geq 0$$

$$\implies E[Y] - E[X] \geq 0 \text{ by linearity}$$

$$\implies E[X] \leq E[Y].$$

In particular, since

$$-|X| \leq X \leq |X|,$$ we have that $|E[X]| \leq E[|X|].$

aka Generalized Triangle Inequality
Suppose $X \notin Y$ are independent simple RVs. Then

$$E[XY] = E[X] \cdot E[Y].$$

Why?

$$X = \sum_{i=1}^{n} x_i 1_{A_i} \quad \text{and} \quad Y = \sum_{j=1}^{m} y_j 1_{B_j},$$

$\{A_i\}$ and $\{B_j\}$ finite partitions of $\Omega$, $\{x_i\}$ and $\{y_j\}$ are distinct, then

$X \notin Y$ are independent iff

$$P(A_i \cap B_j) = P(A_i) P(B_j) \quad \forall i, j.$$  

In that case,

$$E[XY] = \sum_{ij} x_i y_j P(A_i \cap B_j)$$

$$= \sum_{ij} x_i y_j P(A_i) P(B_j)$$

$$= E[X] E[Y].$$

Note: This may be false if $X \notin Y$ not indep!
Example: \( X = \begin{cases} 0 \text{ with prob. } \frac{1}{3} \\ 1 \text{ with prob. } \frac{2}{3} \end{cases} \) e.g. fair coin flip where heads = 1 \( \frac{1}{2} \) tails = -1 (or modified Bernoulli trial)

If \( Y = X \), then

\[ E[X] = E[Y] = 1\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) = 0 \]

**BUT** \( E[XY] = 1 \).

\[
\left[ (1)(1)(\frac{1}{2}) + (-1)(-1)(\frac{1}{2}) = 1 \right]
\]

Example:

We may have \( E[XY] = E[X]E[Y] \) even if \( X \parallel Y \) are NOT independent.

\[
X = \begin{cases} 0 \text{ w/prob } \frac{1}{3} \\ 1 \text{ w/prob } \frac{2}{3} \\ 2 \end{cases}
\]

\[
Y(w) = \begin{cases} 1 & \text{if } X(w) = 0 \text{ or } 2 \\ 5 & \text{if } X(w) = 1 \end{cases}
\]

Compute \( E[XY], E[X], E[Y] \) to verify claim above.