Eliminating the moment conditions

def: A collection of random variables $\{X_\alpha\}_{\alpha \in I}$ are identically distributed if for any Borel measurable function $f: \mathbb{R} \to \mathbb{R}$, the expected value $E[f(X_\alpha)]$ does not depend on $\alpha$, i.e. is the same for all $\alpha \in I$.

Remark: It follows that $\{X_\alpha\}_{\alpha \in I}$ are identically distributed iff $\forall x \in \mathbb{R}$, $P(X_\alpha \leq x)$ does not depend on $\alpha$.

def: A collection of RVs $\{X_\alpha\}_{\alpha \in I}$ are i.i.d. if they are independent and identically distributed.

Thm [Strong LLNs Version 2]: Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each having finite mean $\mu$. Then

$$P\left(\lim_{n \to \infty} \frac{1}{n} (X_1 + \cdots + X_n) = \mu\right) = 1.$$ 

i.e. $S_n \xrightarrow{\text{a.s.}} \mu$ where $S_n = X_1 + \cdots + X_n$.
Theorem: To begin, we assume that $X_i \geq 0$. If not, we can consider separately $X^+$ and $X^-$. We set

$$Y_i = X_i \mathbb{1}_{\{X_i \leq i\}}$$

i.e. $Y_i = X_i$ unless $X_i$ exceeds $i$ (in which case $Y_i = 0$).

Then since $0 \leq Y_i \leq i$, we have that

$$E[Y_i^k] \leq i^k < \infty.$$

Also, note that the $\{Y_i\}$ are independent (function of $X_i$'s). Furthermore, since the $X_i$'s are i.i.d.

$$E[Y_i] = E[X_i \mathbb{1}_{\{X_i \leq i\}}] = E[X_i \mathbb{1}_{\{X_i \leq i\}}] \Rightarrow E[X_i] = \mu$$

as $i \to \infty$, by the MCT.

Now set $S_n = \sum_{i=1}^{\infty} X_i$ and $S_n^k = \sum_{i=1}^{\infty} Y_i$. We compute

$$\text{Var}(S_n^k) = \text{Var}(Y_1) + \cdots + \text{Var}(Y_n) \quad \text{since} \quad \{Y_i\} \text{ indep}$$

$$\leq E[Y_1^2] + \cdots + E[Y_n^2] \quad \text{always true}$$

$$= E[X_1^2 \mathbb{1}_{\{X_1 \leq 1\}}] + \cdots + E[X_n^2 \mathbb{1}_{\{X_n \leq n\}}] \quad \text{by def}$$

$$\leq n \cdot E[X_i^2 \mathbb{1}_{\{X_i \leq n\}}] \quad \text{why?} \quad E[Y_i^k] \leq i^k < \infty$$

$$\leq n \cdot n^2 = n^3 < \infty$$

With $\frac{1}{n} S_n \Rightarrow \mu$, $\frac{1}{n} S_n^k \Rightarrow \mu$. 

\[ \text{Why?} \]
We now choose $\alpha > 1$ (later we will let $\alpha > 1$) and set $u_n = L \alpha^n$. Then $u_n \leq \alpha^n$. Furthermore, since $\alpha^n > 1$, it follows that $u_n \geq \frac{\alpha^n}{2} \Rightarrow \frac{1}{u_n} \leq \frac{2}{\alpha^n}$. Hence, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{u_n} \leq \sum_{n=1}^{\infty} \frac{1}{\alpha^n} \leq \sum_{n=1}^{\infty} \frac{2}{\alpha^n} = \sum_{k=\log_\alpha x}^{\infty} \frac{2}{\alpha^k} = \frac{2/\epsilon}{1 - \frac{1}{\alpha}} \quad (5.4.6)$$

Now (heart of the proof), for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P \left( \left| \frac{S_n^* - E[S_n^*]}{u_n} \right| > \epsilon \right)$$

$$= \sum_{n=1}^{\infty} P \left( \left| \frac{S_n^*}{u_n} - E \left[ \frac{S_n^*}{u_n} \right] \right| > \epsilon \right)$$

$$\leq \sum_{n=1}^{\infty} \frac{\text{Var} \left( \frac{S_n^*}{u_n} \right)}{\epsilon^2} \quad \text{by Chebyshev's Ineq.}$$

$$= \sum_{n=1}^{\infty} \frac{\text{Var} \left( S_n^* \right)}{u_n^2 \epsilon^2} \quad \text{by property of variance}$$

$$\left[ \text{Var}(aX) = a^2 \text{Var}(X) \right]$$
\[ \leq \sum_{n=1}^{\infty} \frac{u_n E[X_1^2 \mathbf{1}_{|X_1| \leq u_n^2}]}{u_n^2 \varepsilon^2} \quad \text{by (5.4.5)} \]

\[ = \frac{1}{\varepsilon^2} E \left[ X_1^2 \sum_{n=1}^{\infty} \frac{1}{u_n} \mathbf{1}_{u_n \leq X_1} \right] \quad \text{by linearity of } E(\cdot) \]

\[ \leq \frac{1}{\varepsilon^2} E \left[ X_1^2 \left( \frac{2/X_1}{1-\frac{1}{\alpha}} \right) \right] \quad \text{by (5.4.6)} \]

\[ = \frac{2}{\varepsilon^2 (1-\frac{1}{\alpha})} E \left[ X_1^2 \cdot \frac{1}{X_1} \right] = \frac{2}{\varepsilon^2 (1-\frac{1}{\alpha})} E[X_1] \]

\[ = \frac{2\mu}{\varepsilon^2 (1-\frac{1}{\alpha})} < \infty. \quad \text{(since } E[X_1] \text{ is assumed to be finite)} \]

\[ \forall \varepsilon > 0, \text{ if } \sum_{n} P(\left| Z_n - Z \right| > \varepsilon) < \infty \text{, then } Z_n \overset{a.s.}{\to} Z \]

This finiteness is key!

It now follows from Cor 5.2.2.

that \[ \left\{ \frac{S_{n^*} - E[S_{n^*}]}{u_n} \right\} \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty. \quad (5.4.7) \]

To complete the proof, need to replace \( \frac{E[S_{n^*}]}{u_n} \) by \( \mu \), replace \( S_{n^*} \) by \( S_n \), and finally replace the index \( u_n \) by the general index \( k \).
First, since \( E[Y_i] \to \mu \) as \( i \to \infty \) and since \( u_n \to \infty \) as \( n \to \infty \), it follows immediately that

\[
\frac{E[S_n^*]}{u_n} \to \mu \text{ as } n \to \infty.
\]

Then by (5.4.7), it follows that \( \left\{ \frac{S_n^*}{u_n} \right\} \overset{a.s.}{\longrightarrow} \mu \) as \( n \to \infty \).

(5.4.8)

Note that

\[
\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} P(X_k > k) \leq \sum_{k=1}^{\infty} P(X_k \geq k)
\]

\[
= \sum_{k=1}^{\infty} P(X_1 \geq k) = E[LX_1,1] \leq E[X_1] = \mu < \infty.
\]

Since \( X_i \)'s are iid by the prop.

from Lec 9

(4.2.9 in \( R \))

By Borel-Cantelli,

\( P(X_k \neq Y_k \text{ i.o.}) = 0 \) so that \( P(X_k = Y_k \text{ a.a.}) = 1 \),

almost always

It follows that, w.p. 1 (i.e. a.s.), as \( n \to \infty \)

\( \) all but a finite set

the limit of \( \frac{S_n^*}{u_n} \) coincides with limit of \( \frac{S_n}{u_n} \).

Hence, (5.4.8) \Rightarrow \left\{ \frac{S_n}{u_n} \right\} \overset{a.s.}{\longrightarrow} \mu
Finally, for an arbitrary index $k$, we can find $n = n_k$ s.t. $u_n \leq k < u_{n+1}$. But then

$$\frac{u_n}{u_{n+1}} \cdot \frac{S_n}{u_n} = \frac{S_n}{u_{n+1}} \leq \frac{S_k}{k} \leq \frac{S_{n+1}}{u_n} = \frac{S_{n+1}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n}.$$ 

Now as $k \to \infty$, we have $n = n_k \to \infty$ so that

$$\frac{u_n}{u_{n+1}} \to \frac{1}{\alpha} \quad \text{and} \quad \frac{u_{n+1}}{u_n} \to \alpha.$$ 

Hence, for any $\alpha > 1$ and $s > 0$, with prob. 1 we have

$$\frac{\mu}{(1+s)\alpha} \leq \frac{S_k}{k} \leq (1+s)\alpha \mu$$

for all sufficiently large $k$. For any $\varepsilon > 0$, choosing $\alpha > 1$ and $s > 0$ s.t. $\frac{\mu}{(1+s)\alpha} > \mu - \varepsilon$ and $(1+s)\alpha \mu < \mu + \varepsilon$

$$\Rightarrow P\left(\left|\frac{S_k}{K} - \mu\right| \geq \varepsilon \text{ i.o.}\right) = 0.$$ 

Hence, by Lemma (5.2.1), we have that as $k \to \infty$,

$$\frac{S_k}{K} \quad \text{a.s.} \to \mu \quad \text{as required.} \quad \Box$$
Now by Prop 5.2.3 (Rosenthal) \( Z_n \xrightarrow{a.s.} Z \Rightarrow Z_n \xrightarrow{p} Z \)

we immediately get a weaker version of LLNs.

* Cor [Weak LLNs version 2]:

Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables, each having finite mean \( \mu \). Then \( \forall \varepsilon > 0, \)

\[
\lim_{n \to \infty} P \left( \left| \frac{1}{n} \left( X_1 + \ldots + X_n \right) - \mu \right| \geq \varepsilon \right) = 0
\]

i.e. \( \frac{S_n}{n} \xrightarrow{p} \mu \) (convergence in probability).

Pf: SLLNs version 2 combined with Prop 5.2.3.