Recall: A Markov chain is a sequence of RVs $X_0, X_1, X_2, \ldots$ taking values in $S$ (sample space) s.t.

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n (= i))$$

$$= P(X_{n+1} = j \mid X_n = i) = P_{ij}$$

for every $n \in \mathbb{N}$ and every sequence $i_0, i_1, \ldots, i_n \in S$ for which $P(X_0 = i_0, \ldots, X_n = i_n) > 0$.

Also,

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \alpha_{i_0} P_{i_0 i_1} \ldots P_{i_{n-1} i_n}$$

Follows that

$$P(X_1 = j) = \sum_{i \in S} P(X_0 = i, X_1 = j)$$

$$= \sum_{i \in S} \alpha_i P_{ij}.$$
Chain rule for conditional probabilities:

\[ P(x_0 = i_0, x_1 = i_1, x_2 = i_2) = P(x_0 = i_0) \cdot P(x_1 = i_1 | x_0 = i_0) \cdot P(x_2 = i_2 | x_0 = i_0, x_1 = i_1) = \alpha_{i_0} \cdot P_{i_0} \cdot P_{i_1} \]
Examples

1. Simple Random Walk on a finite set:

Suppose $S = \{0, 1, \ldots, n\}$

Fix $a \in S$ and let $x_a = 1$ and $x_i = 0$ for $i \neq a$ (i.e.,)

Fix $p \in \mathbb{R}$ s.t. $0 < p < 1$ and let

$$
\begin{align*}
    P_i(t+1) &= p \\
    P_i(t) &= 1 - p 
\end{align*}
$$

for $0 < i < n$

$$P_0 = P_n = 1 \quad \text{absorbing states 0 \& n}$$

(once a particle enters an absorbing state, it cannot leave it)

In matrix form

\[
P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1-p & 0 & p & \cdots & 0 \\
0 & 1-p & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1-p & 0 & p \\
0 & \cdots & 0 & 1-p & 0 \\
\end{pmatrix}
\]

This MC starts at a \& then at each step either

- increases by 1 w/ prob. $p$
- decreases by 1 w/ prob. $1-p$

until reaching state 0 or $n$ (absorbing states).
Unrestricted Random Walk - same setup as gambling game discussed lastlec.

Let $S = \mathbb{Z}$. Fix $a \in \mathbb{Z}$ and let $\alpha_a = 1$, $\alpha_i = 0$ for $i \neq a$.

Let $P_{i,i+1} = p$ and $P_{i,i-1} = 1-p \ orall i \in \mathbb{Z}$, $0 < p < 1$.

This MC has no absorbing states; the particle is free to move anywhere (in increments of $\pm 1$).

Random walk is symmetric if $p = \frac{1}{2}$.

E.g. $X_0 = 1$, $X_1 = 2$, $X_2 = 1$, $X_3 = 0$, $X_4 = -1$, ...

MC on 3 points: $\{1, 2, 3\}$

$S = \{1, 2, 3\}$

$$P = \begin{pmatrix}
1 & 2 & 3 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2}
\end{pmatrix}$$

Check: Sum of "out" arrows for each state is 1.

e.g. $X_0 = 2$
$X_1 = 3$
$X_2 = 1$
$X_3 = 3$
$X_4 = 3$
Ehrenfest's Urn

Consider 2 urns:

- Suppose there are \( d \) balls divided between the 2 urns.
- At each step, choose 1 ball uniformly at random (from all \( d \) balls) and switch it to the opposite urn.

Let \( X_n = \# \) of balls in urn 1 at time \( n \).

Then \( S = \{1, 2, \ldots, d\} \) with

\[
\begin{align*}
P(i, i+1) &= \frac{d-i}{d} \\ P(i, i-1) &= \frac{i}{d}
\end{align*}
\]

For \( 0 \leq i \leq d \)

\[
(P_{i,j} = 0 \text{ if } j \neq i \pm 1)
\]

One might expect that, if \( d \) is large \( \frac{d}{2} \) MC is run for a long time, that there would most likely be \( \approx \frac{d}{2} \) balls in Urn 1.

* We'll consider such cases in a bit!
**Existence Theorem**

**Thm 8.1.1**: Given a non-empty countable set $S$ and non-negative numbers $\{\alpha_i\}_{i \in S}$ and $\{p_{ij}\}_{i, j \in S}$ s.t.

\[ \sum_{i \in S} \alpha_i = 1 \quad \text{and} \quad \sum_{j \in S} p_{ij} = 1 \quad \text{for each} \ i \in S, \ 	ext{f on some prob. space} (\Omega, \mathcal{F}, \mathbb{P}) \] a Markov chain $X_0, X_1, X_2, \ldots$ with initial probabilities $\alpha_i$ and transition probabilities $p_{ij}$.

**Pf.** See Billingsley or Rosenbluth.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be Lebesgue measure on $[0,1]$. Careful partitioning of $[0,1]$ by induction...

**Transience, Recurrence, Irreducibility**

Fundamental notions related to MCs. For simplicity, assume $\alpha_i > 0 \ \forall i \in S$.

**Notation**: $P_i(A)$ means $P(A \mid X_0 = i)$

$E_i(A)$ means expected value w.r.t. $P_i$

**def.** Let $f_{ij}^{(n)} = P_i(X_n = j, \text{but } X_m \neq j \text{ for } 1 \leq m \leq n-1)$ for $i, j \in S$ and $n \in \mathbb{N}$. This is the probability, starting from $i$, that we first hit $j$ at time $n$. 