2 main results give conditions under which it is true that \( E[X_n] \to E[X] \) as \( n \to \infty \)

- **Monotone Convergence Thm**
  
  \( (R \text{ Thm 4.2.2}) \quad E[X_i] \to -\infty \implies \{X_n\} \not\to \{X\} \ldots \)

- **Bounded Convergence Thm**
  
  \( (R \text{ Thm 7.3.1}) \quad \exists K \in \mathbb{R} \text{ s.t. } |X_n| \leq K \quad \forall n \in \mathbb{N} \ldots \)

Now we will establish 2 more similar limit thms:

- **Dominated Convergence Thm**

- **Uniformly Integrable Convergence Thm**

First, another result need to prove DCT above.

**Thm 9.1.1 [Fatou's Lemma]**: If \( X_n \geq C \quad \forall n \in \mathbb{N} \) and some constant \( C \geq -\infty \), then

\[
E \left[ \liminf_{n \to \infty} X_n \right] \leq \liminf_{n \to \infty} E[X_n]
\]

(Allow possibility that both sides are \( \infty \)).

**Pf:** Let \( Y_n = \inf_{k \geq n} X_k \quad \text{and} \quad Y = \lim_{n \to \infty} Y_n \quad = \liminf_{n \to \infty} X_n \).

\( \left( Y_1 = \inf_{k \geq 1} X_k \quad , \quad Y_2 = \inf_{k \geq 2} X_k \quad , \ldots \right) \)
Then \( Y_n \geq C \) (since \( X_n > C \) \( \forall n \) by assumption) and \( \{Y_n\} \not\to Y \). Also, \( Y_n \leq X_n \) (by defn of \( Y \)). By order-preserving property \( \Rightarrow \) MCT, it follows that

\[
\lim_{n \to \infty} E[Y_n] = E[Y] \quad \text{(by MCT)}
\]

\[
\liminf_{n \to \infty} E[Y_n] \quad \text{since the limit exists} \AND \quad E[X_n] \geq E[Y_n] \quad \text{(order-preserving)}
\]

\[
\implies \liminf_{n \to \infty} E[X_n] \geq \liminf_{n \to \infty} E[Y_n] = E[Y]
\]

\[
= E[\liminf_{n \to \infty} X_n]
\]

by def of \( Y \). \( \square \)

Note: \( \liminf_{n \to \infty} X_n \) is interpreted pointwise

ie. its value at \( \omega \) is \( \liminf_{n \to \infty} X_n(\omega) \).

\textbf{Thm 9.1.2 [Dominated Convergence Thm]:} If \( X_1, X_2, \ldots \) are RVs, and if \( X_n \to X \) with prob. 1, and if \( Y \) a RV \( Y \) s.t. \( |X_n| \leq Y \ \forall n \in \mathbb{N} \) and \( E[Y] < \infty \), then

\[
\lim_{n \to \infty} E[X_n] = E[X].
\]
Pf: First note that $Y + X_n \geq 0$. Apply Fatou's Lemma to $\{Y + X_n\}$, we see that

\[ E[Y] + E[X] = E[Y + X] \leq \liminf_n E[Y + X_n] = E[Y] + \liminf_n E[X_n] \]

Since $E[Y] < \infty$, it follows that (cancel $E[Y]$ terms)

\[ E[X] \leq \liminf_n E[X_n]. \]

Similarly, $Y - X_n \geq 0$. Apply Fatou's Lem. to $\{Y - X_n\}$:

\[ E[Y] - E[X] \leq E[Y] + \liminf_n E[-X_n] \]

\[ = E[Y] - \limsup_n E[X_n] \]

\[ \Rightarrow E[X] \geq \limsup_n E[X_n]. \]

However, we always have $\limsup_n E[X_n] \geq \liminf_n E[X_n]$.

Thus, combining

with $\limsup_n E[X_n] \leq E[X] \leq \liminf_n E[X_n]$

\[ \Rightarrow \limsup_n E[X_n] = \liminf_n E[X_n] = E[X]. \]

\[ \lim_{n \to \infty} E[X_n] \]
NOTE: If RV $Y$ is constant, then DCT reduces to BCT.

**Def:** A collection $\{X_n\}$ of random variables is
uniformly integrable if $\lim_{n \to \infty} \sup_{\alpha \geq 0} E[|X_n| 1_{|X_n| \geq \alpha}] = 0$.

Note: Uniform integrability $\Rightarrow$ boundedness of certain expectations

**Prop 9.1.5:** If $\{X_n\}$ is uniformly integrable, then
$\sup_n E[|X_n|] < \infty$. Furthermore, if also $X_n \xrightarrow{a.s.} X$ as $n \to \infty$,
then $E[|X|] < \infty$.

**Thm 9.1.6 [Uniform Integrability Convergence Thm]:**

If $X_1, X_2, \ldots$ are RVs, and if $X_n \to X$ with prob. 1 as $n \to \infty$;
and if $\{X_n\}$ are uniformly integrable, then
$\lim_{n \to \infty} E[X_n] = E[X]$.

**PF (Idea):** Let $Y_n = |X_n - X|$ so that $Y_n \to 0$ as $n \to \infty$.
Show that $E[Y_n] \to 0$ as $n \to \infty$ by considering $Y_n$ in 2 pieces: $Y_n = Y_n 1_{Y_n < \alpha} + Y_n 1_{Y_n \geq \alpha}$. 
it follows from
Then, a triangle inequality that
\[ |E[X_n] - E[X]| \leq E[Y_n] \to 0 \quad \text{as} \quad n \to \infty \quad \text{which proves}
the theorem.

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**Moment Generating Functions & Large Deviations**

(Ref: R § 9 am't, B § 9) - skip 9.2 & 9.4

An interesting connection w/ SLLNs is to estimate
the rate at which \( \frac{S_n}{n} \) converges to the mean \( \mu \).

Proof of SLLN used upper bounds for probabilities
\[ P\left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) \quad \text{for large} \ \varepsilon. \]

Accurate upper & lower bounds for these probabilities
lead to the Law of the Iterated Logarithm, a
theorem giving precise rates for \( \frac{S_n}{n} \to \mu \).

First, we'd like to estimate the prob. of large
deviations from the mean
\[ \to \text{requires use of moment generating functions.} \]
Large Deviations

In particular, if $X_1, X_2, \ldots$ are i.i.d. with common mean $\mu$ and finite variance $\sigma^2$ (i.e. $E[X_i] = \mu \forall i$, $\text{Var}[X_i] = \sigma^2 \forall i$), then by Chebyshev's Ineq.

\[
\forall \epsilon > 0, \quad P\left( \frac{X_1 + \ldots + X_n}{n} \geq \mu + \epsilon \right) \leq \frac{\sigma^2}{n \epsilon} \rightarrow 0 \quad \text{as } n \rightarrow \infty
\]

Q. How quickly is this limit reached?

Does the probability decrease as $O\left(\frac{1}{n}\right)$, or faster?

A. If the MGFs are finite in a neighborhood of 0, then the convergence is actually exponentially fast!

Thm 9.3.4: Suppose $X_1, X_2, \ldots$ are i.i.d. with common mean $\mu$ & $M_{X_i}(s) < \infty$ for $-a < s < b$ where $a, b > 0$.

Then

\[
P\left( \frac{X_1 + \ldots + X_n}{n} \geq \mu + \epsilon \right) \leq \rho^n, \quad n \in \mathbb{N}
\]

where $\rho = \inf_{0 < s < b} \left( e^{-s(\mu + \epsilon)} M_{X_i}(s) \right) < 1$. 
- This theorem gives an exponentially small upper bound on the prob. that the average of the $X_i$'s exceeds its mean by at least $\varepsilon$.

- This is a simple example of a **large deviations** result.

Recall definition of moment generating function of a RV $X$:

$$M_X(s) = E[e^{sX}], \ s \in \mathbb{R}$$

**Some properties**

- $X, Y$ indep $\Rightarrow M_{X+Y}(s) = M_X(s) \cdot M_Y(s)$

- $M_X(0) = 1$

- If $M_X(s) < \infty$ for $|s| < s_0$ for some $s_0 > 0$ (i.e., finite in a neighborhood of 0), then $E|X|^\gamma < \infty$ for all $\gamma < s_0$

  \( M_X(s) \) is **analytic** for $|s| < s_0$ with

  $$M_X(s) = \sum_{n=0}^{\infty} \frac{E[X^n] s^n}{n!}$$

  In particular, the $r$-th derivative at $s=0$ is $E[X^r]$. 
Before we prove Thm 9.3.4, need a lemma.

**Lemma 9.3.5**: Let \( Z \) be a RV with \( E[Z] < 0 \) st. 
\( M_Z(s) < \infty \) for \(-a < s < b\), for some \( a, b > 0\). Then 
\[ P(Z > 0) \leq \rho \text{ where } \rho = \inf_{0 < s < b} M_Z(s) < 1. \]

**Pf**: For any \( s \in (0, b) \) the function \( \phi(x) = e^{sx} \) is increasing, so by Markov's Ineq.,
\[ P(Z > 0) = P(e^{sZ} > 1) \leq \frac{E[e^{sZ}]}{1} = M_Z(s). \]

Now take infimum over \( 0 < s < b \):
\[ P(Z > 0) \leq \inf_{0 < s < b} M_Z(s) = \rho. \]

Lastly, since \( M_Z(0) = 1 \) and \( M_Z'(0) = E[Z] < 0 \) (by assumption), it follows that \( M_Z(s) < 1 \) for all positive \( s \) sufficiently close to \( 0 \), i.e. \( \rho < 1 \). (calculus)

**Pf of Thm 9.3.4**: Let \( Y_i = X_i - \mu - \varepsilon \) st. \( E[Y_i] = -\varepsilon < 0 \).

For \(-a < s < b\), we have that 
\[ M_{Y_i}(s) = E[e^{sY_i}] = e^{-s(\mu + \varepsilon)}. \quad E[e^{sX_i}] = e^{-s(\mu + \varepsilon)} M_{X_i}(s) < \infty \]
(since \( M_{X_i}(s) \) is finite).
Then by Lemma 9.3.5,

\[
P\left(\frac{X_1 + \cdots + X_n}{n} \geq \mu + \varepsilon\right) = P\left(\frac{Y_1 + \cdots + Y_n}{n} \geq 0\right)
\]

\[
= P\left(Y_1 + \cdots + Y_n \geq 0\right)
\]

\[
\leq \inf_{0 < s < b} M_{Y_1 + \cdots + Y_n}(s) = \inf_{0 < s < b} (M_{Y_1}(s))^n = \rho^n
\]

since \(Y_i's\)
are indep

where \(\rho = \inf_{0 < s < b} M_{Y_1}(s) = \inf_{0 < s < b} (e^{-s(\mu + \varepsilon)} M_{X_1}(s)).\)

Lastly, \(\rho < 1\) by Lemma 9.3.5.