Since $\epsilon > 0$ is arbitrary, the "only if" part is proved.

Next consider the "if" part. Since $C_0(\mathbb{R}) \subset C_b(\mathbb{R})$, (2.9) and Theorem 9.2.2 imply that $\mu_n \Rightarrow^v \mu$. As noted in Remark 9.2.1, if $(\mu_n)_{n \geq 1}$ are probability measures then $\mu_n \Rightarrow^d \mu$. So the proof is complete. 

**Definition 9.2.2:**

(a) A sequence of probability measures $\{\mu_n\}_{n \geq 1}$ on $(\mathbb{R}, B(\mathbb{R}))$ is called **tight** if for any $\epsilon > 0$, there exists $M = M_{\epsilon} \in (0, \infty)$ such that

$$\sup_{n \geq 1} \mu_n([-M, M]^c) < \epsilon. \quad (2.10)$$

(b) A sequence of random variables $\{X_n\}_{n \geq 1}$ is called **tight** (or **stochastically bounded**) if the sequence of probability distributions $\{\mu_n\}_{n \geq 1}$ of $\{X_n\}_{n \geq 1}$ is tight, i.e., given any $\epsilon > 0$, there exists $M = M_{\epsilon} \in (0, \infty)$ such that

$$\sup_{n \geq 1} P(|X_n| > M) < \epsilon. \quad (2.11)$$

**Remark 9.2.3:** In Definition 9.2.2 (b), the random variables $X_n$, $n \geq 1$ need not be defined on a common probability space. If $X_n$ is defined on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$, $n \geq 1$, then (2.11) needs to be replaced by

$$\sup_{n \geq 1} P_n(|X_n| > M) < \epsilon.$$

**Example 9.2.3:** Let $X_n \sim \text{Uniform}(n, n+1)$. Then, for any given $M \in (0, \infty)$,

$$P(|X_n| > M) \geq P(X_n > M) = 1 \quad \text{for all } \quad n > M.$$ 

Consequently, for any $M \in (0, \infty)$,

$$\sup_{n \geq 1} P(|X_n| > M) = 1$$

and the sequence $\{X_n\}_{n \geq 1}$ cannot be stochastically bounded.

\textit{not tight}
Example: Let \( \mu_n = S_{n \text{mod} 3} \) — point mass at \( n \text{mod} 3 \).

i.e. \( \mu_1 = S_1 \)
\( \mu_2 = S_2 \) \{ repeats \}
\( \mu_3 = S_0 \)
\( \mu_4 = S_1 \)
\( \mu_5 = S_2 \)
\( \mu_6 = S_0 \)

Q. Is \( \mathcal{E} \mu_n \) tight?

Yes. Take \([a, b] = [0, 3]\).

Then \( \forall \varepsilon > 0 \) \& \( \forall n \in \mathbb{N} \),

\[ \mu_n([0,3]) = S_n([0,3])_{\text{mod} 3} = \left\lceil \frac{[0,3](n)}{3} \right\rceil_{n=0,1,2} = 1 > 1 - \varepsilon. \]

Example: For \( n \geq 1 \), let \( X_n \sim \text{Uniform}(a_n, 2 + a_n) \) where \( a_n = (-1)^n \). Then

\[ X_1 = U(-1, 2 - 1 = 1) \quad \text{so} \quad |X_1| \leq 1 \]
\[ X_2 = U(1, 2 + 1 = 3) \quad \text{so} \quad |X_2| \leq 3 \]
\[ X_3 = U(-1, 1) \]
\[ X_4 = U(1, 3) \]

\[ \Rightarrow |X_n| \leq 3 \quad \forall n \in \mathbb{N} \]

\[ \Rightarrow \sup_{n \geq 1} P(|X_n| > 3) < \varepsilon \quad \forall \varepsilon > 0 \]

\[ \Rightarrow \text{seq. of prob. distr.'s of } X_n \text{ is tight} \]

However, \( \{X_n^2\}_{n \geq 1} \) does not converge in distribution to a RV X.
Properties

- Any finite collection of probability measures is tight.
- Union of 2 tight collections of prob. meas. is tight.
- Any sub-collection of a tight collection is tight.

Thm 11.1.10: If $\{\mu_n\}$ is a tight sequence of probability measures, then there is a subsequence $\{\mu_{n_k}\}$ of a prob. measure $\mu$ s.t. $\mu_{n_k} \Rightarrow \mu$ as $n \to \infty$.

i.e. $\{\mu_{n_k}\}$ converges weakly to $\mu$

PF Idea: By Helly Selection Principle, $F_{n_k} \to F$ s.t.

$F_{n_k}(x) \to F(x)$ at all continuity points of $F$.

Using tightness, can show that $F$ is actually a prob. distribution function (i.e. in particular $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$, as desired).

Cor 11.1.11: Let $\{\mu_n\}$ be a tight seq. of prob. dist's on $\mathbb{R}$. Suppose that $\mu$ is the only possible weak limit of $\{\mu_n\}$, meaning $\mu_{n_k} \Rightarrow \nu$ implies that $\nu = \mu$.

Then $\mu_n \Rightarrow \mu$ as $n \to \infty$. 
One last result: sufficient condition for a sequence of measures to be tight

Lemma 11.1.13: Let \( \{\mu_n\} \) be a sequence of prob. measures on \( \mathbb{R} \) with characteristic functions \( \phi_n(t) = \int e^{itx} \, d\mu_n(x) \).
Suppose \( \exists \) function \( g \) (continuous at 0) s.t. \( \lim_{n \to \infty} \phi_n(t) = g(t) \) for each \( |t| < t_0 \) for some \( t_0 > 0 \). Then \( \{\mu_n\} \) is tight.

# Theorem 11.1.14 [Continuity Thm]: Let \( \mu, \mu_1, \mu_2, \ldots \) be prob. measures with corresp. characteristic functions \( \phi, \phi_1, \phi_2, \ldots \).
Then \( \mu_n \Rightarrow \mu \) iff \( \phi_n(t) \to \phi(t) \) \( \forall t \in \mathbb{R} \).

In words: prob. measures converge weakly to \( \mu \) iff their char. functions converge pointwise to that of \( \mu \).

PF: First suppose that \( \mu_n \Rightarrow \mu \) as \( n \to \infty \). Then since \( \cos(tx) \) & \( \sin(tx) \) are bounded continuous functions, we have that

\[
\phi_n(t) = \int \cos(tx) \, d\mu_n(x) + i \int \sin(tx) \, d\mu_n(x)
\]

\( \to \)

\[
\int \cos(tx) \, d\mu(x) + i \int \sin(tx) \, d\mu(x) \quad \text{by def of weak con}
\]

\( = \phi(x) \) as \( n \to \infty \) for each \( t \in \mathbb{R} \).
Conversely, suppose that $\phi_n(t) \to \phi(t)$ for each $t \in \mathbb{R}$. Then by using $g = \phi$ in Lemma 11.1.13, the $\mathcal{E}_n$ are tight. Now suppose that $\mu_n \Rightarrow \nu$ for some subseq $\mathcal{E}_{\mu_n}$ of some measure $\nu$. Then

$$\phi_{\mu_n}(t) \to \phi_{\nu}(t) \quad \text{as } t \in \mathbb{R} \text{ where } \phi_{\nu}(t) = \int e^{itx} d\nu(x).$$

On the other hand, we know that (by assumption)

$$\phi_{n_k}(t) \to \phi(t) \quad \text{as } t \in \mathbb{R}.$$ 

Hence, $\phi_{\nu} = \phi$. By Fourier uniqueness (Cor 11.1.7), this implies that $\nu = \mu$.

Thus, $\mu$ is the only possible weak limit of $\mathcal{E}_{\mu_n}$, so by Cor 11.1.11, it follows that $\mu_n \Rightarrow \mu$ as $n \to \infty$. 

\[\square\]
The Central Limit Theorem

(Ref: Rosenthal § 11.2, Billingsley § 27)

Last time we proved the Continuity Theorem (11.1.4), so we are now in a position to prove the classical Central Limit Thm (CLT).

First, compute the characteristic function for the standard normal distribution $N(0,1)$, i.e.

for RV $X \sim N(0,1)$, density $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ w.r.t. Lebesgue measure.

$$
\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x) \, dx = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx
$$

Turns out that

$$
\phi_X(t) = M_X(it) = e^{+(it)^2/2} = e^{-t^2/2} \quad \forall \, t \in \mathbb{R}
$$

(can justify this using complex analysis)

Prop. If $X \sim N(0,1)$, then $\phi_X(t) = e^{-t^2/2} \quad \forall \, t \in \mathbb{R}$. 
Thm 11.2.2 [Central Limit Thm]: Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean $m$ and finite variance $v$. Set $S_n = X_1 + \cdots + X_n$. Then as $n \to \infty$,

$$\mathcal{L} \left( \frac{S_n - nm}{\sqrt{vn}} \right) \Rightarrow \mu_N \quad \text{where } \mu_N = N(0,1)$$

(in words, when i.i.d. RVs are added, their properly normalized sum tends toward a normal distribution as $n$ gets large.)

**PF:** By replacing $X_i$ by $\frac{X_i - m}{\sqrt{v}} \ orall i$, we can assume that $m = 0 \neq v = 1$.

Let $\phi_n(t) = E \left[ e^{it \frac{S_n}{\sqrt{vn}}} \right]$ be the characteristic function of $\frac{S_n}{\sqrt{vn}}$ (by definition). By the Continuity Theorem & Proposition above, it suffices to show that $\lim_{n \to \infty} \phi_n(t) = \phi(t) = e^{-t^2/2}$ for each fixed $t \in \mathbb{R}$.

More details:

We want to show that the sequence $\mu_n = \mathcal{L} \left( \frac{S_n}{\sqrt{vn}} \right)$ (recalling that $m = 0$ and $v = 1$)
converges weakly to $\mu_N = N(0,1)$. By Continuity Thm,

$$\mu_n \Rightarrow N(0,1) \iff \phi_n(t) \to e^{-t^2/2} \text{ for } t \in \mathbb{R}.$$ 

$\phi(t)$ by Prop.

for $N(0,1)$
To do this, set $\phi_x(t) = E[e^{itX_1}]$. Then as $n \to \infty$, use a Taylor expansion and Prop 11.0.1 to get

$$\phi_x^{(j)}(0) = i^j E[X^j]$$

$$\phi_n(t) = E\left[e^{it\left(X_1 + \ldots + X_n\right)/\sqrt{n}}\right]$$

$$= E\left[e^{i\left(\frac{t}{\sqrt{n}}\right)X_1} e^{i\left(\frac{t}{\sqrt{n}}\right)X_2} \ldots e^{i\left(\frac{t}{\sqrt{n}}\right)X_n}\right]$$

$$= E\left[e^{i\left(\frac{t}{\sqrt{n}}\right)X_1}\right]^n \text{ since } X_i's \text{ are i.i.d.}$$

$$= \left(\phi_x\left(t\frac{1}{\sqrt{n}}\right)\right)^n$$

$$= \left(1 + \frac{it}{\sqrt{n}} E[X_1] + \frac{1}{2!} \left(\frac{it}{\sqrt{n}}\right)^2 \frac{E[X_1^2]}{\sqrt{n}} + o\left(\frac{1}{n}\right)\right)^n$$

$$= \left(1 + \frac{i^2 t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n$$

$$= \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n$$

since $i^2 = -1$

$$\to e^{-t^2/2} \text{ as } n \to \infty$$, as claimed.

\[\text{Note that } o\left(\frac{1}{n}\right) \text{ means a quantity } q_n \text{ s.t. } \frac{q_n}{\sqrt{n}} \to 0 \text{ as } n \to \infty, \text{ "little o" notation}\]