Martingales (cont')

Recall Def: A stochastic process \( \{X_n : n = 0, 1, 2, \ldots\} \)

is a martingale if \( E[|X_n|] < \infty \ \forall n \) and w.p. 1

\[
E[X_{n+1} \mid X_0, X_1, \ldots, X_n] = X_n
\]

\( \forall_n \in \sigma(X_0, \ldots, X_n) \)

Example 1: Unbiased random walk on \( \mathbb{Z} \) where \( X_0 = 0 \),

\( P_i, i+1 = P_i, i-1 = \frac{1}{2} \ \forall i \in \mathbb{Z} \). (Actually true in any
dimension \( \mathbb{Z}^d \))

Example 2: Let \( Z_1, Z_2, \ldots \) be independent RVs. Let

\( X_0 = 0 \) \( \forall \ X_n = Z_1 + \cdots + Z_n \) for \( n = 1, 2, \ldots \) (or
\( X_n = \sum_{i=1}^{n} Z_i \))

Then \( X_{n+1} = X_n + Z_{n+1} \).

\[
E[X_{n+1} \mid X_0, \ldots, X_n] = E[X_n \mid X_0, \ldots, X_n] + E[Z_{n+1} \mid X_0, \ldots, X_n]
\]

\[
= X_n + E[Z_{n+1}]
\]

since \( Z_{n+1} \) is

Since \( X_n \in \sigma(X_0, \ldots, X_n) \) indep of \( X_0, \ldots, X_n \)

Thus, \( \{X_n\} \) is a martingale iff

\( E[Z_1] = E[Z_2] = \cdots = 0 \).
Example 3: Let $Z_1, Z_2, \ldots$ be indep. RVs. Let $X_n = \sum_{i=1}^{n} Z_i \neq Y_n = e^{X_n}$. Then $Y_{n+1} = e^{X_{n+1}} = e^{X_n + Z_{n+1}} = Y_n e^{Z_{n+1}}$.

$$E[Y_{n+1} | Y_0, \ldots, Y_n] = E[Y_n e^{Z_{n+1}} | Y_0, \ldots, Y_n]$$

$$= Y_n E[e^{Z_{n+1}} | Y_0, \ldots, Y_n]$$

$$= Y_n E[e^{Z_{n+1}}] \quad \text{since } Z_{n+1} \text{ is indep of } Y_0, \ldots, Y_n$$

Thus, $\{Y_n\}$ is a martingale iff

$$E[e^{Z_1}] = E[e^{Z_2}] = \cdots = 1.$$

def: A stochastic process $\{X_n\}$ is a **submartingale** if $E[|X_n|] < \infty \forall n$ and

$$E[X_{n+1} | X_0, \ldots, X_n] \geq X_n.$$

def: Stoch. process $\{X_n\}$ is a **supermartingale** if $E[|X_n|] < \infty \forall n$ and

$$E[X_{n+1} | X_0, \ldots, X_n] \leq X_n.$$

* These names are arguably the reverse of what they should be intuitively.
A martingale \{X_n\} can be thought of as the fortune at time \( n \) of a player who is betting on a fair game.

- Submartingale - as outcome of betting on a favorable game
- Supermartingale - as outcome of betting on an unfavorable game

Result 1: You cannot make money betting on martingales! In particular, if you choose to stop playing at some bounded time \( N \), then expected winnings \( E[X_N] = E[X_0] \).

Initial fortune

To explain this, suppose \{X_n\} is a submartingale.

\[
E[X_{n+1} | X_0, \ldots, X_n] \geq X_n
\]

\[
\Rightarrow E\left[ E[X_{n+1} | X_0, \ldots, X_n] \right] \geq E[X_n]
\]

\[
\Rightarrow E[X_{n+1}] \geq E[X_n] \quad \text{by induction}
\]

\[
E[X_n] \geq E[X_{n-1}] \geq \ldots \geq E[X_1] \geq E[X_0]
\]
So if \( \{X_n\} \) is a martingale, then
\[
E[X_n] = E[X_{n-1}] = \ldots = E[X_0] \quad \text{or} \quad E[X_n] = E[X_0]
\]
\( \forall n \in \mathbb{N} \).

Q. What about \( E[X_\tau] \) where \( \tau \) is a random time?

Is it still true that \( E[X_\tau] = E[X_0] \)?

Ans: No, in general.

**def:** A random variable \( \tau \) (which takes on values \( 0, 1, 2, \ldots \)) is called a stopping time if the event \( \{ \tau = n \} \) only depends on \( X_0, \ldots, X_n \) and not on any future information (say \( X_{n+1} \) for instance).

- Hitting time problems

**Example 1:** \( \tau = \min \{ n : X_n > 1 \} \) is a stopping time.

\( \{ \tau = n \} \) means that \( X_0 \leq 1, X_1 \leq 1, \ldots, X_{n-1} \leq 1 \) and \( X_n > 1 \), which only depends on \( X_0, \ldots, X_n \).

**Example 2:** \( \tau = \max \{ n : X_n > 1 \} \) is NOT a stopping time. Consider \( \{ \tau = 1 \} \). This means that \( X_1 > 1 \) and \( X_2 \leq 1 \), but this depends on future values of \( X_n \).

\( \text{Last exit problems} \)

(last time a process hits a state or set of states)
Thm [Optional Stopping Theorem]: Consider a stopping time $T$ that is bounded above by a constant $N$ s.t. $T \leq N$. If $\{X_n\}$ is a martingale, then
\[ E[X_T] = E[X_N] = E[X_0]. \]

Martingale Convergence

If $\{X_n\}$ is a martingale (or submartingale), will it converge a.s. to some random value?

Ans: In general, no.

  e.g. Simple symmetric RW is a martingale, but it is null recurrent and has no stationary dist'n.

Result 2: Concerns submartingales (stochastic analogues of non-decreasing sequences): If they are bounded above, they converge to a limit a.s.

\[ \rightarrow \text{Known as the Martingale Convergence Thm} \]
Thm [Martingale Convergence]: Let $\{X_n\}$ be a submartingale. Suppose that $\sup_n E[|X_n|] < \infty$. Then there is a finite RV $X$ s.t.

$$X_n \xrightarrow{a.s.} X.$$ 

Example: Markov chain $\{X_n\}$ on non-negative integers,

$X_0 = 50$, transition probabilities $p_{ij} = \frac{1}{2i+1}$, $0 \leq j \leq 2i$ and $p_{ij} = 0$ o.w.

That is, if $X_n = i$ then $X_{n+1} \sim \text{Uniform on } \{0, 1, \ldots, 2i\}$.

$$\begin{cases} 
  X_0 = 50 = i \\
  p_{50,j} = \frac{1}{2(50)+1} = \frac{1}{101} \quad \text{for } 0 \leq j \leq 100, \frac{1}{2} \text{ so on.}
\end{cases}$$

This MC is a Martingale (exercise to reader to check!) and hence also a submartingale.

It follows that $X_n \xrightarrow{a.s.} 0$. 