The Extension Theorem
(Ref: R §2.3, B §3)

Last time: Probability Measure Spaces
aka Probability Triples

\[ \Omega \text{ countable:} \]
\[ (\Omega, \mathcal{F}, P) \text{ straightforward} \]
\[ \text{Power Set} \]
\[ \text{e.g. } \Omega = \{1, 2, 3\} \]

\[ \Omega \text{ uncountable:} \]
\[ (\Omega, \mathcal{F}, P) \text{ complicated} \]
\[ \text{Not the Power Set} \]
\[ \text{e.g. } \Omega = [0,1] \]

Big Picture via \( U(0,1) \) example

The Extension Thm allows us to automatically construct valid probability triples for \( \Omega = [0,1] \).

If once a prob. measure is constructed on a semi-algebra \( \mathcal{S} \),
then it can be extended to a \( \sigma \)-algebra \( \mathcal{M} \).

Recall def: \( \mathcal{S} = \{ \text{all intervals contained in } [0,1] \} \)

\[ \uparrow \]

all open, closed, half open, singleton
intervals \( \subseteq [0,1] \neq \phi \)
Q. How to create a $\sigma$-algebra?

Recall that

- $B_0 = \{\text{all finite unions of elements of } J\}$
- $B_1 = \{\text{all finite or countable unions of elements of } J\}$

both are not $\sigma$-algebras

What we want is:

- $B = \sigma(J) =$ Borel $\sigma$-algebra

the $\sigma$-algebra generated by $J$ (the set of all open (or closed) intervals in $[0,1]$).

- $M =$ Lebesgue measurable sets, also a $\sigma$-algebra.

Note that $B \subseteq M$. (power set)

Also note that $B$ is generally not equal to $2^\mathcal{O}$.

Example of a non-Borel set: Vitali Set
(non-measurable set)

$M$ contains more sets than $B$ (generally)

$M$ is a complete measure space, $B$ is not complete.

i.e. every subset of a null set is measurable (having measure 0)
Thm 2.3.1 [The Extension Thm]: Let \( J \) be a semialgebra of subsets of \( \Omega \). Let \( P: J \to [0,1] \) with \( P(\emptyset) = 0 \) and \( P(\Omega) = 1 \), satisfying finite superadditivity:

\[
P\left( \bigcup_{i=1}^{k} A_i \right) \geq \sum_{i=1}^{k} P(A_i) \quad \text{whenever } A_1, \ldots, A_k \in J, \quad \bigcup_{i=1}^{k} A_i \in J, \text{ and } \{A_i\} \text{ are disjoint,}
\]

and also countable monotonicity:

\[
P(A) \leq \sum_i P(A_i) \quad \text{for } A, A_1, A_2, \ldots \in J \text{ with } A \leq \bigcup_i A_i .
\]

Then there is a \( \sigma \)-algebra \( M \supseteq J \) and a countably additive probability measure \( P^* \) on \( M \) s.t.

\[
P^*(A) = P(A) \quad \forall A \in J.
\]

(That is, \((\Omega, M, P^*)\) is a valid prob. triple which agrees with our previous probabilities on \( J \)).

i.e. For an interval \( I \in J \), \( P(I) = \text{length of interval} \)

\[
I = [a, b] \Rightarrow P(I) = b - a
\]

Note: Conclusions of this Thm \( \implies \) \( \Box \) must actually hold with equality.

However, we only need inequality to apply Thm.
This theorem provides a way to construct (complicated) prob. triples on a full σ-algebra, using only probabilities defined on the much simpler subsets (e.g. intervals) in J.

Q. How to prove this Thm?

**Key Idea**: Outer measure $P^*$ defined by

$$P^*(A) = \inf_{A_i, A_2, \ldots \in J} \sum_{i} \leq P(A_i) \quad \text{for} \quad A \subseteq \Omega$$

In other words, define $P^*(A)$ for any subset $A \subseteq \Omega$ to be the infimum of sums of $P(A_i)$, where $\{A_i\}$ is any countable collection of elements from the original semialgebra $J$ whose union contains $A$.

Use the values of $P(A)$ for $A \in J$ to help us define $P^*(A)$ for any $A \subseteq \Omega$.

We know that $P^*$ will not necessarily be a proper prob. measure $\forall A \subseteq \Omega$ (recall prop. from Lecture 1).

existence of non measurable sets
However, it is still useful that $P^*(A)$ is at least defined $\forall A \in \Omega$.

* We will show that $P^*$ is a probability measure on some $\sigma$-algebra $\mathcal{M}$, and that $P^*$ is an extension of $P$.

Properties of $P^*$ (outer measure)

- $P^*(\emptyset) = 0$ (take $A_i = \emptyset \ \forall i$)

- Monotone Property holds: $A \subseteq B \Rightarrow P^*(A) \leq P^*(B)$

Lemma 1: $P^*$ is an extension of $P$, i.e.

$$P^*(A) = P(A) \ \forall A \in \mathcal{J}.$$  

Proof: Let $A \in \mathcal{J}$. It follows from ** in the Extension Thm that $P^*(A) \geq P(A)$ (since $P(A) = \sum_{i} P(A \cap A_i) \leq \sum_{i} P(A_i)$), where $A_i \in \mathcal{J}$ and $A = \bigcup A_i$.

On the other hand, choosing $A_1 = A$ and $A_i = \emptyset$ for $i \geq 2$ in def. of $P^*$

$$\Rightarrow P^*(A) \leq P(A). \ \text{Thus, } P^*(A) = P(A) \ \forall A \in \mathcal{J}.$$
Lemma 2: $P^*$ is countably subadditive, i.e.

$$P^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} P^*(B_n) \text{ for any } B_1, B_2, \ldots \subseteq \Omega.$$ 

Pf: See Rosenthal.

Now set

$$M = \{A \subseteq \Omega : P^*(A \cap E) + P^*(A^c \cap E) = P^*(E) \quad \forall \ E \subseteq \Omega\}$$

set of all subsets $A$ with property that $P^*$ is additive on the union of $A \cap E$ with $A^c \cap E$ $\forall E \subseteq \Omega$

By subadditivity, we always have

$$P^*(A \cap E) + P^*(A^c \cap E) \geq P^*(E)$$

So the def of $M$ above is equivalent to

$$M = \{A \subseteq \Omega : P^*(A \cap E) + P^*(A^c \cap E) \geq P^*(E) \quad \forall \ E \subseteq \Omega\}$$

(sometimes helpful!)

Next, we will show that

$p^*$ is countably additive on $M.$
Lemma 3: If $A_1, A_2, \ldots \in M$ are disjoint, then

$$P^*(\bigcup_n A_n) = \sum_n P^*(A_n).$$

Pf: Suppose $A_1 \neq A_2$ are disjoint, and $A_1 \in M$. Then

$$P^*(A_1 \cup A_2) = P^*(A_1 \cap (A_1 \cup A_2)) + P^*(A_1^c \cap (A_1 \cup A_2))$$

since $A_1 \in M$

$$= P^*(A_1) + P^*(A_2)$$ since $A_1 \neq A_2$ are disjoint.

By induction, this lemma holds for any finite set $\{A_i\}$. With countably many disjoint $A_i \in M$, we see that for any $m \in \mathbb{N}$,

$$\sum_{n \leq m} P^*(A_n) = P^*(\bigcup_{n \leq m} A_n) \leq P^*(\bigcup_n A_n),$$

by monotonicity.

Since this is true for any $m \in \mathbb{N}$, it follows that

$$\sum_n P^*(A_n) \leq P^*(\bigcup_n A_n).$$

On the other hand, by subadditivity we have

$$\sum_n P^*(A_n) \geq P^*(\bigcup_n A_n).$$

Thus, lemma holds for countably many $A_i$ as well.
Big Picture once again:

Lemmas 1, 3, 7, 8 (last 2 still to be shown)

\[ \Rightarrow M \text{ is a } \sigma\text{-algebra containing } J, \]
\[ p^* \text{ is a probability measure on } M, \]
\[ p^* \text{ is an extension of } P. \]

This proves the Extension Theorem.

Apply the Extension Thm to conclude the following:

Thm 2.4.4: There exists a probability triple \((\Omega, M, p^*)\)

such that \(\Omega = [0,1] \), \(M\) contains all intervals in \([0,1]\), and
for any interval \(I \subseteq [0,1]\), \(p^*(I)\) is the length of \(I\).

- The probability measure is \text{Lebesgue measure} on \([0,1]\).
- \(M\) is a \(\sigma\)-algebra that contains \(B\), the
  \text{Borel } \sigma\text{-algebra} of subsets of \([0,1]\).

Note: \(J \subseteq B \subseteq M\)

Moving forward, \((\Omega, B, P)\)

we will consider \(\sigma\)-algebra

uncountable \text{Lebesgue measure}
on \(B\)