Due on Thursday February 14 at the beginning of lecture.

1. Let $(\Omega_1, \mathcal{F}_1, P_1)$ be Lebesgue measure on $[0, 1]$. Consider a second probability triple $(\Omega_2, \mathcal{F}_2, P_2)$ defined as follows: $\Omega_2 = \{1, 2\}$, $\mathcal{F}_2$ consists of all subsets of $\Omega_2$, and $P_2$ is defined by $P_2(\{1\}) = \frac{1}{3}$, $P_2(\{2\}) = \frac{2}{3}$, and additivity. Let $(\Omega, \mathcal{F}, P)$ be the product measure of $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ (see definition on p.23 of Rosenthal).

(a) Express each of $\Omega$, $\mathcal{F}$, and $P$ as explicitly as possible.

(b) Find a set $A \in \mathcal{F}$ such that $P(A) = \frac{3}{4}$.

(c) Give an example of a random variable $X$ defined on $(\Omega_1, \mathcal{F}_1, P_1)$ (other than a uniform RV). Verify that $X$ is $\mathcal{F}_1$-measurable.

(d) Give an example of a random variable $Y$ defined on $(\Omega_2, \mathcal{F}_2, P_2)$. Verify that $Y$ is $\mathcal{F}_2$-measurable.

2. Show that if events $E$ and $F$ are independent and $E \subseteq F$, then either $P(E) = 0$ or $P(F) = 1$.

3. Prove that if $E$ and $F$ are independent events, then so are $E$ and $F^C$.

4. Suppose that $A_1, A_2, \ldots, A_n$ are mutually independent events.

(a) Prove that $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ and $A_n$ are independent events. [Hint: Use induction!]

(b) Prove that $P(\bigcap_{k=1}^n A_k^c) = \prod_{k=1}^n P(A_k^c)$. [Hint: Use induction, part (a), and problem 3.]

5. Suppose that $\{A_n\} \nearrow A$. Let $f : \Omega \to \mathbb{R}$ be any function. Prove that

$$\lim_{n \to \infty} \inf_{\omega \in A_n} f(\omega) = \inf_{\omega \in A} f(\omega).$$