Due on Tuesday February 26 at the beginning of lecture. Note: R stands for exercises in Rosenthal’s book.

1. (R 3.6.7)
Let $(\Omega, F, P)$ be the uniform distribution on $\Omega = \{1, 2, 3\}$. That is, $P(A) = |A|/|\Omega|$ for all $A \in F = 2^\Omega$ (where $|A|$ is the cardinality of set $A$). Give an example of a sequence $A_1, A_2, \cdots \in F$ such that

$$P\left(\liminf_{n \to \infty} A_n\right) < \liminf_{n \to \infty} P(A_n) < \limsup_{n \to \infty} P(A_n) < P\left(\limsup_{n \to \infty} A_n\right),$$

i.e. such that all three inequalities are strict.

2. (R 3.6.9)
Let $(\Omega, F, P)$ be a probability space and let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be events in $F$.

(a) Prove that

$$\left(\limsup_{n \to \infty} A_n\right) \cap \left(\limsup_{n \to \infty} B_n\right) \supseteq \limsup_{n \to \infty} (A_n \cap B_n).$$

(b) Give an example where the above inclusion is strict, and another example where it holds with equality.

3. (R 3.6.11)
Let $\{X_n\}_{n=1}^\infty$ be independent random variables with $X_n \sim \text{Uniform}\{\{1, 2, \ldots, n\}\}$ (see definition in Problem 1 above). Compute $P(X_n = 5 \text{ i.o.})$, the probability that an infinite number of the $X_n$ are equal to 5.

4. (R 3.6.16)
Consider infinite, independent, fair coin tossing (see lecture 7 notes). Let $H_n$ be the event that the $n^{th}$ coin toss is heads. Determine the following probabilities:

(a) $P(H_{n+1} \cap H_{n+2} \cap \cdots \cap H_{n+9} \text{ i.o.}).$

(b) $P(H_{n+1} \cap H_{n+2} \cap \cdots \cap H_{2n} \text{ i.o.}).$

(c) Prove that $P(H_{n+1} \cap H_{n+2} \cap \cdots \cap H_{n+[\log_2 n]} \text{ i.o.})$ must equal either 0 or 1.

5. (R 3.6.13)
Let $X_1, X_2, \ldots$ be defined jointly on some probability space $(\Omega, F, P)$, with $E[X_n] = 0$ and $E[(X_n)^2] = 1$ for all $n \in \mathbb{N}$. Prove that $P(X_n \geq n \text{ i.o.}) = 0.$
Let \((\Omega, \mathcal{F}, P)\) be Lebesgue measure on \([0, 1]\) and set

\[
X(\omega) = \begin{cases} 
1, & 0 \leq \omega < \frac{1}{4} \\
2\omega^2, & \frac{1}{4} \leq \omega < \frac{3}{4} \\
\omega^2, & \frac{3}{4} \leq \omega \leq 1.
\end{cases}
\]

Compute \(P(X \in A)\) where

(a) \(A = [0, 1]\).

(b) \(A = [\frac{1}{2}, 1]\).