Due on Thursday February 1 at the beginning of lecture.

1. Suppose that $\Omega = \{1, 2\}$, with $P(\emptyset) = 0$ and $P(\{1, 2\}) = 1$. Suppose $P(\{1\}) = \frac{1}{4}$. Prove that $P$ is countably additive if and only if $P(\{2\}) = \frac{3}{4}$.

2. Suppose that $\Omega = \{1, 2, 3\}$ and $\mathcal{F}$ is the collection of all subsets of $\Omega$.
   
   (a) List all the elements in $\mathcal{F}$.
   
   (b) Find (with proof) necessary and sufficient conditions on the real numbers $x$, $y$, and $z$ such that there exists a countably additive probability measure $P$ on $\mathcal{F}$, with $x = P(\{1, 2\})$, $y = P(\{2, 3\})$, and $z = P(\{1, 3\})$.

3. Suppose that $\Omega = \mathbb{N}$ is the set of natural numbers, and $P$ is defined for all $A \subseteq \Omega$ by $P(A) = 0$ if $A$ is finite, and $P(A) = 1$ if $A$ is infinite. Is $P$ countably additive? Justify your answer.

4. Prove that
   
   $$\mathcal{B}_0 = \{\text{all finite unions of elements of } \mathcal{J}\}$$

   is an algebra (or field) of subsets of $\Omega = [0, 1]$, meaning that it contains $\Omega$ and $\emptyset$, and is closed under the formation of complements and of finite unions and intersections. Recall that
   
   $$\mathcal{J} = \{\text{all intervals contained in } [0, 1]\}.$$

5. Let $\Omega = \{1, 2, 3, 4\}$. Determine whether or not each of the following is a $\sigma$-algebra.
   
   (a) $\mathcal{F}_1 = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$
   
   (b) $\mathcal{F}_2 = \{\emptyset, \{3\}, \{4\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$
   
   (c) $\mathcal{F}_3 = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}$

6. Prove that $\mathcal{B}_0$ (defined in Problem 4 above) is not a $\sigma$-algebra.