Due on Thursday February 22 at the beginning of lecture. Note: R stands for exercises in Rosenthal’s book.

1. (R 3.6.7)
   Let \((\Omega, F, P)\) be the uniform distribution on \(\Omega = \{1, 2, 3\}\). That is, \(P(A) = |A|/|\Omega|\) for all \(A \in F = 2^\Omega\) (where \(|A|\) is the cardinality of set \(A\)). Give an example of a sequence \(A_1, A_2, \ldots \in F\) such that
   \[P\left(\liminf_n A_n\right) < \liminf_n P(A_n) < \limsup_n P(A_n) < P\left(\limsup_n A_n\right),\]
   i.e. such that all three inequalities are strict.

2. (R 3.6.9)
   Let \((\Omega, F, P)\) be a probability space and let \(A_1, A_2, \ldots, B_1, B_2, \ldots\) be events in \(F\).
   (a) Prove that
   \[\left(\limsup_n A_n\right) \cap \left(\limsup_n B_n\right) \supseteq \limsup_n (A_n \cap B_n).\]
   (b) Give an example where the above inclusion is strict, and another example where it holds with equality.

3. (R 3.6.11)
   Let \(\{X_n\}_{n=1}^\infty\) be independent random variables with \(X_n \sim \text{Uniform}\{1, 2, \ldots, n\}\) (see definition in Problem 1 above). Compute \(P(X_n = 5 \ i.o.)\), the probability that an infinite number of the \(X_n\) are equal to 5.

4. (R 3.6.16)
   Consider infinite, independent, fair coin tossing (see lecture 7 notes). Let \(H_n\) be the event that the \(n^{th}\) coin toss is heads. Determine the following probabilities:
   (a) \(P(H_{n+1} \cap H_{n+2} \cap \cdots \cap H_{n+9} \ i.o.).\)
   (b) \(P(H_{n+1} \cap H_{n+2} \cap \cdots \cap H_{2n} \ i.o.).\)
   (c) \(P(H_{n+1} \cap H_{n+2} \cap \cdots \cap H_{n+[2\log_2 n]} \ i.o.).\)

5. (R 3.6.13)
   Let \(X_1, X_2, \ldots\) be defined jointly on some probability space \((\Omega, F, P)\), with \(E[X_n] = 0\) and \(E[(X_n)^2] = 1\) for all \(n \in \mathbb{N}\). Prove that \(P(X_n \geq n \ i.o.) = 0.\)

6. (R 4.5.2)
   Let \(X\) be a random variable with finite mean, and let \(a \in \mathbb{R}\) be any real number. Prove that
   \[E\left[\max(X, a)\right] \geq \max(E[X], a).\]