1. \( Y \sim \text{Weibull distribution } W(\alpha, \beta). \)

The survival function is given by

\[
S_Y(y) = Pr(Y \geq y) = e^{-\left(\frac{y}{\alpha}\right)^\beta}, \quad y \geq 0, \quad \alpha, \beta > 0.
\]

(a) **Cumulative distribution and its inverse functions.**

The cumulative distribution function (CDF) is derived as follows:

\[
F_Y(y) = P(Y \leq y) = 1 - P(Y > y) = 1 - S_Y(y) = 1 - e^{-\left(\frac{y}{\alpha}\right)^\beta}, \quad y \geq 0, \quad \alpha, \beta > 0.
\]

The inverse CDF is obtained setting the CDF equal to \( u \) and solving \( y \) as follows:

\[
F_Y(y) = x \\
1 - e^{-\left(\frac{y}{\alpha}\right)^\beta} = x \\
e^{-\left(\frac{y}{\alpha}\right)^\beta} = 1 - x \\
\left(\frac{y}{\alpha}\right)^\beta = -\ln(1 - x) = \ln\left[\frac{1}{1 - x}\right] \\
\frac{y}{\alpha} = \ln\left(\frac{1}{1 - x}\right)^\frac{1}{\beta} \\
y = \alpha \left[\ln\left(\frac{1}{1 - x}\right)\right]^\frac{1}{\beta}.
\]

Thus,

\[
F_Y^{-1}(x) = y = \alpha \left[\ln\left(\frac{1}{1 - x}\right)\right]^\frac{1}{\beta}, \quad x \in (0, 1).
\]

(b)

\[
Y = F_Y^{-1}(U) = \alpha \left[\ln\left(\frac{1}{1 - U}\right)\right]^\frac{1}{\beta}, \quad \text{where } U \sim \in (0, 1).
\]
(c)\

\[ U = F_X(x) = \int_0^x e^{-t} dt = 1 - e^{-x} \]

(d)\

\[ Y = F_Y^{-1}(u) = F_Y^{-1}(F_X(x)) = F_Y^{-1}(1 - e^{-x}) = \alpha \left[ \ln \frac{1}{1 - (1 - e^{-x})} \right]^{\frac{1}{\beta}} \]

\[ = \alpha \left[ \ln (e^x) \right]^{\frac{1}{\beta}} = \alpha (x)^{\frac{1}{\beta}} \]

Thus, \( Y = \alpha (X)^{\frac{1}{\beta}} \) where \( X \sim \text{exp}(1) \)

2. Simulate \( Y \sim W(\alpha, \beta) \) random variates

- First simulate \( n \) standard exponential random variables, \( X \)
- Compute \( Y = \alpha (X)^{\frac{1}{\beta}} \)
- Below is the R code

```r
rweibull <- function(A, B, n) ### A = alpha, B=beta and n=sample size
{
    X <- rexp(n)
    Y <- A * (X^(1/B))
}

W <- rweibull(2, 2, 1000)
# Plot a histogram of the values from Weibull distribution
# with parameter (alpha=2,beta=2) and sample size 1000
hist(W, freq=FALSE, breaks=20,
    main = "Histogram of 1,000 Sample Samples Simulate from Weibull ",
    xlab = "Values from the weibull distribution",
    ylab = "Density of Values", col = "orange")
```
`V <- rweibull(2,2,10000)`

# Plot a histogram of the values from Weibull distribution
# with parameter `(alpha=2, beta=2)` and sample size 1000

`hist(V, freq=FALSE, breaks=20,`
`     main = "Histogram of 10,000 Samples Simulate from Weibull",
    xlab = "Values from the weibull distribution",
    ylab = "Density of Values", col = "light blue")`
3. Markov Chain Simulation

The discrete-time random walk \( \{X_n : n = 0, 1, 2, \ldots\} \) on integer values starting at \( X_0 = 0 \) and the transition probabilities: \( P_{i,i+1} = p \) and \( P_{i,i-1} = 1 - p \)

```r
# Function to simulate a Markov chain.
# Requires the following input: n = number of time steps in sim, P = transition matrix, x1 = initial condition
MC.sim <- function(n,p,x1) {
    sim <- as.numeric(n)
    if (missing(x1)) {
        sim[1] <- 0 # random initial condition
    }
    else {
        sim[1] <- x1
    }
    for (i in 2:n) {
        newstate <- sample(c(-1,1),1,prob=c(1-p,p)) +x1
        sim[i] <- newstate
        x1=newstate
    }
    return(sim)
}
```

```r
count <- 10000
P <- matrix(c(0.5, 0.5, 0.5, 0.5), nrow = 2)
x1 <- 0
sim <- MC.sim(n = count, P = P, x1 = x1)
```
(a) Random walk simulation with transition probability: $P_{i,i+1} = 0.5$

```r
X<-MC.sim(100,0.5,0)
[See R code at end for details on plot below]
```
(b) Random walk simulation with transition probability: $P_{i,i+1} = 0.8$

```r
Y <- MC.sim(100, 0.8, 0)
[See R code at end for details on plot below]
```

The difference in the graph is due to the transition probabilities. In the first graph, the process has equal chance of moving up or down making it change path in any direction. In the second graph, transition probability for upward movement is higher. There is the natural tendency for the process to move upward.
4. Application of Markov Chain

One use of Markov chains in modeling a real-world phenomena in communication system is in the transmission of digits 0 and 1. Each digit transmitted must pass through several stages, at each of which there is a probability $p$ that the digit entered will be unchanged when it leaves. The $\{X_t : t = 0, 1, 2, 3, 4, \ldots\}$ is a two states Markov Chain process with discrete space and discrete time. State space: $S = \{0, 1\}$ and Time: $T = 0, 1, 2, 3, \ldots$. The transition probability is given by $P_{ii} = p$ and $P_{ij} = p$, for $i \neq j$

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$
# STAT 753 Homework 3 solution - simulate a random walk

# Simulate a random walk with transition probability p (move right 1 step), and left 1 step with probability 1-p

```r
MC.sim <- function(n,p,x1) {
  sim <- as.numeric(n)
  if (missing(x1)) {
    sim[1] <- 0  # initial condition: start at state 0
  } else {
    sim[1] <- x1
  }
  for (i in 2:n) {
    newstate <- sample(c(-1,1),1,prob=c(1-p,p)) + x1
    sim[i] <- newstate
    x1 = newstate
  }
  sim
}
```

# Generate 5 sample paths and plot on same figure

```r
p = 0.5  # symmetric case
run <- MC.sim(100,p,1)
plot(run, type="l", col="blue", ylim=c(-18,18), xlab="Time step", ylab="State")
run <- MC.sim(100,p,1)
points(run, type="l", col="red")
run <- MC.sim(100,p,1)
points(run, type="l", col="darkgreen")
run <- MC.sim(100,p,1)
points(run, type="l", col="orange")
run <- MC.sim(100,p,1)
points(run, type="l", col="magenta")
```

# Generate 5 sample paths and plot on same figure

```r
p = 0.8  # asymmetric case
run <- MC.sim(100,p,1)
plot(run, type="l", col="blue", ylim=c(0,70), xlab="Time step", ylab="State")
run <- MC.sim(100,p,1)
points(run, type="l", col="red")
run <- MC.sim(100,p,1)
points(run, type="l", col="darkgreen")
run <- MC.sim(100,p,1)
points(run, type="l", col="orange")
run <- MC.sim(100,p,1)
points(run, type="l", col="magenta")
```