0. Problem 56 in Ch. 5

Events occur independently each day with prob. $p$

$N(n) =$ total # of events that occur on first $n$ days.

$T_r =$ day on which the $r^{th}$ event occurs.

(a) Distribution of $N(n)$?

$N(n) \sim \text{Binomial} (n, p)$

(b) Distribution of $T_1$?

$T_1 \sim \text{geometric} (p)$ $\left( \text{mean of } T_1 \text{ is } \frac{1}{p} \right)$

$P(T_1 = k) = (1-p)^{k-1} p$ for $k=1, 2, 3, ...$

(i.e. discrete version of exponential waiting time)

(c) Distribution of $T_r$?

$T_r \sim \text{Negative Binomial} (r, p)$

$P(T_r = k) = \binom{k-1}{r-1} (1-p)^{k-r} p^r$, $k=r, r+1, ...$

(i.e. sum of $r$ i.i.d. geometric$(p)$ RVs)

(discrete version of gamma waiting time)
(d) Given \( N(n) = r \), show that the set of \( r \) days on which events occurred has the same distribution as a random selection (w/o replacement) of \( r \) of the values \( 1, 2, \ldots, n \).

\[
P(\text{events at } i_1, i_2, \ldots, i_r \mid N(n) = r) = \frac{P(\text{events at } i_1, \ldots, i_r, N(n) = r)}{P(N(n) = r)}
\]

**Numerator:** Probability of a particular sequence of events where successes are at positions \( i_1, i_2, \ldots, i_r \) and failures are at positions \( \{1, 2, \ldots, n\} - \{i_1, i_2, \ldots, i_r\} \) for the total of \( n \) events. This is:

\[
p^r (1-p)^{n-r}
\]

**Denominator:** All possible combinations of \( r \) successes in \( n \) events:

\[
\binom{n}{r} p^r (1-p)^{n-r}
\]

\[
\Rightarrow \quad \odot = \frac{p^r (1-p)^{n-r}}{\binom{n}{r} p^r (1-p)^{n-r}} = \frac{1}{\binom{n}{r}}
\]
2. Problem 57, Ch 5

Events occur according to a Poisson process with rate $\lambda = 2$ per hour. Let $N(t)$ denote the Poisson Process.

(a) Probability of no events between 8 pm \(\rightarrow\) 9 pm?

$$P(N(9) - N(8) = 0) = P(N(1) = 0) \quad \text{by stat. increments}$$

\[
\frac{9 \text{ pm}}{8 \text{ pm}} = e^{-2(1)} (2(1))^0
\]

\[
= \frac{0!}{e^{-2}} = 0.1353
\]

(b) Starting at noon, what is the expected time at which the 4th event occurs?

$$t_0 = 12 \text{ pm (noon)}$$

Waiting times are i.i.d. exponential with rate $\lambda = 2$ per hour

$$\Rightarrow T_4 \sim \text{gamma} (n=4, \lambda = 2)$$

Waiting time until the 4th event occurs

$$\Rightarrow E[T_4] = \frac{\alpha}{\lambda} = \frac{4}{2} = 2 \text{ hours}$$

$$\Rightarrow t_0 + E[T_4] = 12 \text{ pm} + 2 \text{ hours} = 2 \text{ pm}$$
(c) \[ P(N(8) - N(6) \geq 2) = P(N(2) \geq 2) = 1 - P(N(2) \leq 1) \]

\[ = 1 - P(N(2) = 0) - P(N(2) = 1) \]

\[ = 1 - \frac{e^{-4}(4)^0}{0!} - \frac{e^{-4}(4)^1}{1!} \]

\[ = 1 - e^{-4} - 4e^{-4} \]

\[ = 0.9084 \]
### HW 5 - Solutions (Fall 2017)

#### Problem 3

### Part a

```r
lambda <- function(t) 1 + 5*(1 + sin(pi*t))
curve(lambda, 0, 10, lwd=2, col="purple", main="Intensity Function")
```

Set max time to run the simulation

```r
t_max = 10
```

Run 1

```r
run1 <- NHPP.sim(lambda, t_max)
n1 <- length(run1)-1
```

Run 2

```r
run2 <- NHPP.sim(lambda, t_max)
n2 <- length(run2)-1
```

Run 3

```r
run3 <- NHPP.sim(lambda, t_max)
n3 <- length(run3)-1
```

Find longest sample path and max height

```r
nmax <- max(n1,n2,n3)
hmax <- ceiling(max(run1,run2,run3))
```

Plot the 3 sample paths on the same graph

```r
plot(run1, 0:n1, type="b", pch=19, col="blue", xlim=c(0,hmax), ylim=c(0,nmax), xlab="Time", ylab="N(t)", main="Nonhomogeneous Poisson Process: lambda(t) = a(1+sin(b*pi*t))")
points(run2, 0:n2, type="b", pch=19, col="red")
points(run3, 0:n3, type="b", pch=19, col="aquamarine4")
legend("topleft", c("run 1","run 2","run 3"), pch=c(19,19,19), col=c("blue","red","aquamarine4"))
```

### Part b

Compute average $\lambda$ from the intensity function $\lambda(t)$ defined above

```r
average_lambda <- (integrate(lambda, 0, 10)$value)/10
average_lambda  #should be 6 for this problem
```

Run a HPP with this rate

```r
PP.sim <- function (rate, num.events, num.sims = 1, t0 = 0)
{
  if (num.sims == 1) {
    x = t0 + cumsum(rexp(n = num.events, rate = rate))
    return(c(t0,x))
  }
  else {
    xtemp = matrix(rexp(n = num.events * num.sims, rate = rate), num.events)
    x = t0 + apply(xtemp, 2, cumsum)
    return(rbind(rep(t0, num.sims), x))
  }
}
```

Run the function and plot the time series (for 1 simulation)

```r
rate = average_lambda
num.events = 75
run4 <- PP.sim(rate, num.events)
plot(run4, 0:num.events, xlab = "Time", ylab = "Number of Events", main = "Comparison of NHPP and HPP with average rate", type="b", pch = 19, col="blue", ylim=c(0,10))
points(run1, 0:n1, type="b", pch=19, col="red")
legend("topleft", c("HPP with average rate","NHPP"), pch=c(19,19), col=c("blue","red"))
```
4. Given \( E[Y_i] = \mu, \ Var(Y_i) = \sigma^2, \ E[N(t)] = \lambda t, \ Var(N(t)) = \lambda t \).

\[
Z(t) = \sum_{i=1}^{N(t)} Y_i
\]

\[
E[Z(t)] = E[N(t)] \cdot E[Y_i] = \lambda t \cdot \mu
\]

\[
Var(Z(t)) = E[Var(Z(t) | N(t))] + Var(E[X(t) | N(t)])
\]

\[
= E[N(t)] Var(Y_i) + (E[Y_i])^2 Var(N(t))
\]

\[
= \lambda t \cdot \sigma^2 + \mu^2 \cdot \lambda t
\]

\[
= \lambda t (\sigma^2 + \mu^2)
\]