Best Linear Unbiased Estimator (BLUE)

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Outline

• Definition of BLUE.
• Unbiased linear estimator.
• Derivation of BLUE.
BLUE Definition

- **MVUE**: Minimum variance unbiased estimator.
- **Unbiased estimator**: correct on average.
- Minimize the error variance for each of the $n$ parameters (most efficient linear estimator).

$$J = \min E \left\{ \| \theta - \hat{\theta} \|^2 \right\}$$ (unbiased)  

i.e. unbiased and efficient **linear** estimator (may or may not be MVUE).

$$z(k) = H(k)\theta + \nu(k)$$

$$\hat{\theta}_{BLU}(k) = F(k)z(k)$$
Unbiasedness Condition

For deterministic measurement matrix and linear measurement model, an estimator of $\theta$ is unbiased if

$$F(k)H(k) = I_n, \forall k$$

Note: $n^2$ equality constraints
Proof

\[ z(k) = H(k)\theta + \nu(k) \]

\[ \hat{\theta}(k) = F(k)z(k) = F(k)[H(k)\theta + \nu(k)] \]

\[ E\{\hat{\theta}(k)\} = F(k)H(k)E\{\theta\} + 0 = I_n \theta \]

for an unbiased estimator.

\[ F(k)H(k) = I_n \]
BLUE is a WLSE

**Theorem:** The BLUE of $\theta$ is a WLSE with $W(k) = R^{-1}(k)$, where $R(k)$ is the covariance matrix of the noise process $\nu(k)$.

\[
R(k) = E\{\nu(k)\nu(k)^T\}
\]

\[
H^T(k)R^{-1}(k)H(k)\widehat{\theta}_{BLU}(k) = H^T(k)R^{-1}(k)z(k)
\]
Proof: BLUE Variance

• Optimization: Minimize the sum of the squares of the estimation errors, subject to constraints.

\[ J = \min_{\hat{\theta}} E \left\{ \| \theta - \hat{\theta} \|^2 \right\} = \min_{\hat{\theta}} \sum_{i=1}^{n} E \left\{ (\theta_i - \hat{\theta}_i)^2 \right\} \]

• Consider each parameter separately

• Mean square error of the \( i^{th} \) parameter

\[ \min_{\hat{\theta}_i} E \left\{ (\theta_i - \hat{\theta}_i)^2 \right\}, i = 1, \ldots, n \]
Proof: Unbiased Constraint

- Unbiased Estimator: \( \hat{\theta}(k) = F(k)z(k) \)

\[
F(k)H(k) = I_n = \begin{bmatrix} f_1^T \\ \vdots \\ f_n^T \end{bmatrix} H(k)
\]

\[
f_i^T H(k) = [0_{1 \times (i-1)} \ 1 \ 0_{1 \times (n-i)}] = e_i^T
\]

\( e_i^T = i^{th} \) row of the identity matrix.

\[
\theta_i = e_i^T \theta = f_i^T H(k) \theta, \quad i = 1, \ldots, n
\]
Mean-square Error

• Rewrite objective function as $n$ functions

\[
E \left\{ [\theta_i - \hat{\theta}_{iBLU}(k)]^2 \right\} = E \left\{ [\theta_i - f_i^T z(k)]^2 \right\}
\]

\[
= E \left\{ [\theta_i - f_i^T (H\theta + \nu(k))]^2 \right\}
\]

\[
= E \left\{ [\theta_i - \theta_i - f_i^T \nu(k)]^2 \right\}
\]

\[
= f_i^T E \{\nu(k)\nu^T(k)\} f_i
\]

\[
= f_i^T R(k) f_i, \quad i = 1, ..., n
\]
Optimization

- Minimize the sum of the squares of the mean estimation errors, subject to constraints.
- Include constraints using $n^2$ Lagrange multipliers.

$$J[(f_i, \lambda_i), i = 1, \ldots, n] = \sum_{i=1}^{n} f_i^T R(k) f_i + \sum_{i=1}^{n} \lambda_i^T (H^T(k) f_i - e_i)$$

$$= \sum_{i=1}^{n} [f_i^T R f_i + \lambda_i^T (H^T f_i - e_i)] = \sum_{i=1}^{n} J_i(f_i, \lambda_i)$$
First Necessary Condition

\[ J_i(f_i, \lambda_i) = f_i^T R(k) f_i + \lambda_i^T (H^T(k) f_i - e_i) \]

- Differentiate w.r.t. \( f_i \)

\[ \frac{\partial J_i(f_i, \lambda_i)}{\partial f_i} = 2R(k)f_i + H(k)\lambda_i = 0 \]

\[ f_i = -\frac{1}{2} R^{-1}(k)H(k)\lambda_i \]
Second Necessary Condition

\[ J_i(f_i, \lambda_i) = f_i^T R f_i + \lambda_i^T (H^T f_i - e_i) \]

Differentiate w.r.t. the vector \( \lambda_i \) (assume \( H \) full rank)

\[
\frac{\partial J_i(f_i, \lambda_i)}{\partial \lambda_i} = H^T f_i - e_i = 0
\]

\[
H^T f_i = H^T \left[ -\frac{1}{2} R^{-1} H \lambda_i \right] = e_i
\]

\[
\lambda_i = -2 (H^T R^{-1} H)^{-1} e_i
\]
Eliminate $\lambda_i$

\[ f_i = -\frac{1}{2}R^{-1}H\lambda_i \]

\[ \lambda_i = -2(H^TR^{-1}H)^{-1}e_i \]

\[ f_i = -\frac{1}{2}R^{-1}H \times -2(H^TR^{-1}H)^{-1}e_i \]

\[ f_i = R^{-1}H(H^TR^{-1}H)^{-1}e_i \]
BLUE Solution

\[ f_i = R^{-1}H(H^TR^{-1}H)^{-1}e_i, \; i = 1, ..., n \]

\[ F^T_{BLU} = [f_1 \; f_2 \; \ldots \; f_n] \]

\[ = R^{-1}H(H^TR^{-1}H)^{-1}[e_1 \; e_2 \; \ldots \; e_n] \]

\[ = R^{-1}H(H^TR^{-1}H)^{-1}I_n \]

- Transpose and use \((A^{-1})^T = (A^T)^{-1}\)

\[ F_{BLU} = (H^TR^{-1}H)^{-1}H^TR^{-1} \]
When is BLUE an LSE?

**Theorem:** If the covariance matrix $R(k)$ of the noise process $v(k)$ is given by $R(k) = E\{v(k)v^T(k)\} = \sigma_v^2 I_k$, then the BLUE of $\theta$ is a LS estimator $\hat{\theta}_{BLU}(k) = \hat{\theta}_{LS}(k)$

**Proof**

$\hat{\theta}_{BLU}(k) = \left[ H^T(k)R^{-1}(k)H(k) \right]^{-1} H^T(k)R^{-1}(k)z(k)$

$= \sigma_v^2(k) \left[ H^T(k)H(k) \right]^{-1} \frac{H^T(k)z(k)}{\sigma_v^2(k)} = \hat{\theta}_{LS}(k)$
References
