Discretization of CT Model

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Outline

- Discretize CT systems to use the DKF.
- Use numerical integration for time-varying case.
- Van Loan procedure for numerical computation.

CT State-space Equations

\[
\dot{x}(t) = Fx(t) + Gu(t) \\
x(t) = n \times 1 \text{ state vector} \\
u(t) = m \times 1 \text{ zero-mean input vector}
\]

- Discretize the state equation
  \[
x(k + 1) = \Phi(k + 1, k)x(k) + w(k) \\
E\{w(k)w^T(k)\} = R(k)
\]

Alternative Discrete Model

\[
x(k + 1) = \Phi(k + 1, k)x(k) + \Gamma(k)w(k) \\
E\{w(k)w^T(k)\} = I_n \\
E\{\Gamma(k)w(k)w^T(k)\Gamma^T(k)\} = \Gamma(k)\Gamma^T(k)
\]

- Simpler form for the discrete process noise.
- Equivalent to model without \( \Gamma(k) \)
Solution of State Equation

\[ x(t) = e^{F(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)} G u(\tau) d\tau \]

- State-transition matrix \( \Phi(t, t_0) \), time-varying case.
- Matrix exponential (time-invariant case)

\[ e^{Ft} = \sum_{i=0}^{\infty} \frac{(Ft)^i}{i!} = \Phi^{-1} \{ [sI_n - F]^{-1} \} \]

Matrix Exponential

- Use truncated series.

\[ e^{F\Delta t} \approx \sum_{i=0}^{N} \frac{(F\Delta t)^i}{i!} \]

- Use enough terms to get the desired approximation.
- Acceptable for small sampling period \( \Delta t \).

Response Due to Noise Input

\[ w(k) = x_{zs}(k + 1) = \int_{k\Delta t}^{(k+1)\Delta t} e^{F([k+1]\Delta t - \tau)} G u(\tau) d\tau \]

(i) \( \xi = (k + 1)\Delta t - \tau \)

\[ w(k) = \int_{0}^{\Delta t} e^{F\xi} G u((k + 1)\Delta t - \xi) d\xi \]

(ii) \( \xi = -(k\Delta t - \tau) \)

\[ w(k) = \int_{0}^{\Delta t} e^{F(\Delta t - \xi)} G u(\xi + k\Delta t) d\xi \]

- Used later in a proof.

Mean of Process Noise

\[ w(k) = \int_{0}^{\Delta t} e^{F\xi} G u((k + 1)\Delta t - \xi) d\xi \]

\[ E\{w(k)\} = \int_{0}^{\Delta t} e^{F\xi} G E\{u((k + 1)\Delta t - \xi)\} d\xi \]

\[ m_w = \left[ \int_{0}^{\Delta t} e^{F\xi} G d\xi \right] m_u \]

- Zero-mean \( w(k) \) if \( u(k) \) is zero-mean.
- Gaussian \( w(k) \) if \( u(k) \) is Gaussian (linearity).
- White \( w(k) \) if \( u(k) \) is white (Show with \( E\{w(k)w^T(l)\} = 0, k \neq l \)).
Covariance Matrix

\[ Q(k - l) = E\{w(k)w^T(l)\} \]

\[ = \int_0^{\Delta t} \int_0^{\Delta t} \left[ e^{F\xi}GE\{u((k + 1)\Delta t - \xi) \times \right]d\xi\ d\eta \]

\[ = \int_0^{\Delta t} \int_0^{\Delta t} e^{F\xi}GR_{uu}(\xi - \eta + l - k)G^T e^{F^T\eta}d\xi\ d\eta \]

\[ Q = \int_0^{\Delta t} \int_0^{\Delta t} e^{F\xi}GW\delta(\xi - \eta)G^T e^{F^T\eta}d\xi\ d\eta \]

\[ Q = \int_0^{\Delta t} e^{F\xi}GWG^T e^{F^T\xi}d\xi \]

Alternative Form

\[ w(k) = \int_0^{\Delta t} e^{F(\Delta t - \xi)}Gu(\xi + k\Delta t)d\xi \]

- Similarly, obtain the form
  \[ Q(k) = E\{w(k)w^T(k)\} \]

\[ = \int_0^{\Delta t} e^{F(\Delta t - \tau)}GWG^T e^{F^T(\Delta t - \tau)}d\tau \]

- Used for numerical evaluation.

Example: Integrated Gauss-Markov

Unity Gaussian white noise

\[ X_2(s) \rightarrow \left[ \begin{array}{c} \frac{2\sigma^2\beta}{s + \beta} \\ \frac{1}{s} \end{array} \right] \rightarrow X_1(s) \]

\[ \dot{x}_1(t) = x_2(t) \]

\[ \dot{x}_2(t) = -\beta x_2(t) + \sqrt{2\sigma^2\beta} u(t) \]

\[ x(t) = [x_1 \ x_2]^T \]

\[ \dot{x}(t) = \left[ \begin{array}{cc} 0 & 1 \\ 0 & -\beta \end{array} \right] x(t) + \left[ \begin{array}{c} 0 \\ \sqrt{2\sigma^2\beta} \end{array} \right] u(t) \]

\[ y(t) = [1 \quad 0]x(t) \]

State-transition Matrix

\[ \Phi(t) = \mathcal{L}^{-1}\{[sI_n - F]^{-1}\} = \mathcal{L}^{-1}\left[\begin{bmatrix} \frac{s}{s + \beta} & 0 \\ 0 & 1 \end{bmatrix}\right]^{-1} \]

\[ = \mathcal{L}^{-1}\left[\begin{bmatrix} \frac{s + \beta}{s} & 1 \\ 0 & s(s + \beta) \end{bmatrix}\right] \]

\[ \Phi(\Delta t)G = \left[\begin{array}{cc} 1 & (1 - e^{-\beta\Delta t})/\beta \\ 0 & e^{-\beta\Delta t} \end{array}\right] \left[\begin{bmatrix} 0 \\ \sqrt{2\sigma^2} \end{bmatrix}\right] \]

\[ = \sqrt{2\sigma^2}\beta \left[\begin{bmatrix} 1 - e^{-\beta\Delta t}/\beta \\ e^{-\beta\Delta t} \end{bmatrix}\right] \]
**Discrete Process Noise**

\[ Q = \int_0^{\Delta t} \int_0^{\Delta t} e^{F\xi} GR_{\mu\mu} (\xi - \eta) G^T e^{P\xi} d\xi d\eta \]

\[ = 2\sigma^2 \beta \int_0^{\Delta t} \delta (\xi - \eta) \left( \frac{1 - e^{-\beta \xi}}{\beta} - e^{-\beta \eta} \right) d\xi d\eta \]

\[ = 2\sigma^2 \beta \int_0^{\Delta t} \left[ \frac{1 - e^{-\beta \xi}}{\beta} \right] \left[ \frac{1 - e^{-\beta \xi}}{\beta} - e^{-\beta \xi} \right] d\xi \]

\[ = 2\sigma^2 \beta \int_0^{\Delta t} \left[ \frac{(1 - e^{-\beta \xi})^2}{\beta^2} - e^{-\beta \xi} (1 - e^{-\beta \xi})/\beta \right] d\xi \]

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**Noise Covariance Matrix**

\[ Q = \begin{bmatrix} E\{x_1^2ZS\} & E\{x_1ZSx_2ZS\} \\ E\{x_2ZSx_1ZS\} & E\{x_2^2ZS\} \end{bmatrix} \]

\[ = 2\sigma^2 \int_0^{\Delta t} \left[ \frac{1 - 2e^{-\beta \xi} + e^{-2\beta \xi}}{\beta} \right] \frac{e^{-\beta \xi} - e^{-2\beta \xi}}{\beta} d\xi \]

\[ = 2\sigma^2 \left[ \frac{\left( \xi + \frac{2e^{-\beta \xi} - e^{-2\beta \xi}}{\beta} \right)/\beta - \frac{e^{-\beta \xi} - e^{-2\beta \xi}}{\beta} + \frac{2\beta}{2\beta}}{1 - 2e^{-\beta \Delta t} + e^{-2\beta \Delta t}} \right] \]

\[ = \frac{\sigma^2}{\beta} \left[ (2\beta \Delta t + 4e^{-\beta \Delta t} - e^{-2\beta \Delta t})/\beta - 1 - 2e^{-\beta \Delta t} + e^{-2\beta \Delta t} \right] \]

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**Van Loan Procedure**

1- Form the matrix \( M \)

\[ M = \begin{bmatrix} -F & GWG^T \\ 0 & F^T \end{bmatrix} \]

2- Obtain the matrix exponential \( e^{M\Delta t} \)

\[ e^{M\Delta t} = \begin{bmatrix} e^{-F\Delta t} & e^{-F\Delta t} Q \\ 0 & e^{F^T\Delta t} \end{bmatrix} \]

3- Transpose the lower right corner.

\[ \Phi = e^{F^T\Delta t} = \left( e^{F^T\Delta t} \right)^T \]

4- Calculate \( Q = \Phi \times [\text{upper right corner}] \)

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**Proof**

\[ [sI - M]^{-1} = \begin{bmatrix} sI + F & -GWG^T \\ 0 & sI - F^T \end{bmatrix}^{-1} \]

\[ = \begin{bmatrix} (sI + F)^{-1} & (sI + F)^{-1} GWG^T (sI - F^T)^{-1} \\ 0 & (sI - F^T)^{-1} \end{bmatrix} \]

Inverse Laplace Transform (use slide 10 for \( Q \))

\[ e^{M\Delta t} = \begin{bmatrix} e^{-F\Delta t} & e^{-F(-\Delta t + \Delta t)} \int_0^{\Delta t} e^{-F\tau} GWG^T e^{F^T(\Delta t - \tau)} d\tau \\ 0 & \Phi \end{bmatrix} \]
Example: Harmonic Process

\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \]

\( W = 1, \quad GWG^T = \begin{bmatrix} 0 \\ 2 \end{bmatrix} 1 \begin{bmatrix} 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \)

\( M = \begin{bmatrix} -F \\ 0 \end{bmatrix} GWG^T = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \)

\[
e^{MN} = \begin{bmatrix} 0.995 & -0.0998 \\ 0.0998 & 0.995 \end{bmatrix} \begin{bmatrix} -7 \times 10^{-4} & -0.02 \\ -0.02 & 0.3987 \end{bmatrix}, \quad \Phi = [e^{F\Delta t}]^T = \begin{bmatrix} 0.995 \\ -0.0998 \end{bmatrix} \begin{bmatrix} 0.995 \\ 0.995 \end{bmatrix}
\]

\[
Q = \Phi \times [upper \ right \ corner] = 10^{-4} \begin{bmatrix} 13 \\ 199 \end{bmatrix} \begin{bmatrix} 199 \\ 3987 \end{bmatrix}
\]

Conclusion

- Simple numerical procedure.
- Can use truncated series to calculate the matrix exponential since the sampling period is typically small.