Sequential DKF Computation

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Outline

- Sequential processing of measurements in the Discrete Kalman Filter (DKF).
- Transformation to make measurements uncorrelated and allow sequential processing.

Derivation of DKF

Process and measurement models

$$\boldsymbol{x}_{k+1} = \phi_k \boldsymbol{x}_k + \boldsymbol{w}_k$$
$$\boldsymbol{z}_k = H_k \boldsymbol{x}_k + \boldsymbol{v}_k$$

$\boldsymbol{x}_k = n \times 1$ state vector at t_k

$\phi_k = n \times n$ state-transition matrix at t_k

$\boldsymbol{z}_k = m \times 1$ measurement vector at t_k

$H_k = m \times n$ measurement matrix at t_k

Noise

$\boldsymbol{w}_k = n \times 1$ zero-mean white Gaussian process noise vector at t_k

$\boldsymbol{v}_k = m \times 1$ zero-mean white Gaussian measurement noise vector at t_k

$$E\{\boldsymbol{w}_k \boldsymbol{w}_i^T\} = \begin{cases} Q_k, i = k \\ [\boldsymbol{0}], i \neq k \end{cases}$$

$$E\{\boldsymbol{v}_k \boldsymbol{v}_i^T\} = \begin{cases} R_k, i = k \\ [\boldsymbol{0}], i \neq k \end{cases}$$

$$E\{\boldsymbol{w}_k \boldsymbol{v}_i^T\} = [\boldsymbol{0}]$$

Error Covariance Update

Block diagonal R_k

$$\left(P_k^+\right)^{-1} = \left(P_k^-\right)^{-1} + H_k^T R_k^{-1} H_k = \left(P_k^-\right)^{-1}$$

$$+ \begin{bmatrix} H_k^{1\,T} & H_k^{2\,T} & \dots & H_k^{l\,T} \end{bmatrix} \begin{bmatrix} \left(R_k^1\right)^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \left(R_k^2\right)^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \left(R_k^l\right)^{-1} \end{bmatrix} \begin{bmatrix} H_k^1 \\ H_k^2 \\ \vdots \\ H_k^l \end{bmatrix}$$

$$\left(P_k^+\right)^{-1} = \left(P_k^-\right)^{-1} + \sum_{i=1}^{l} H_k^{i\,T} \left(R_k^i\right)^{-1} H_k^i$$

Software Design

$$\left(P_k^+\right)^{-1} = \left(P_k^-\right)^{-1} + \sum_{i=1}^{l} H_k^{i\,T} \left(R_k^i\right)^{-1} H_k^i$$

Freedom in implementing P^{-1} calculation

- Add all terms with one operation.
- Sequentially add terms.
- Parallel computation of terms then addition.

Information Matrix P^{-1}

- Inverse of uncertainty or error covariance matrix (also see B & H, Section 6.7).
- Add term as each measurement block is processed $\Rightarrow$ increase information.

Sequential Processing

- Reduce vector measurement to a sequence of scalar measurements.
- Inversion of R matrix in error covariance computation reduces to a simple division.
- Uses data transformation based on the modal decomposition of R.
- Transformation leaves noise terms uncorrelated.

Data Transformation

- Modal Decomposition of Covar. Matrix

$$R = L\Lambda L^T, \Lambda = diag\{\lambda_1, \dots, \lambda_m\}$$
$$\lambda_i > 0\ (real), i = 1,2, \dots, m$$
$$L^T \boldsymbol{z} = \bar{\boldsymbol{z}} = L^T H \boldsymbol{x} + L^T \boldsymbol{v} = \bar{H}\boldsymbol{x} + \bar{\boldsymbol{v}}$$
$$E\{\bar{\boldsymbol{v}}_k \bar{\boldsymbol{v}}_k^T\} = L^T E\{\boldsymbol{v}_k \boldsymbol{v}_k^T\} L$$
$$= L^T R L = \Lambda$$

- Uncorrelated $\boldsymbol{v}_{ki}$, measurements processed one at a time.

Error Covariance Summation

$$\left(P_k^+\right)^{-1} = \left(P_k^-\right)^{-1} + H_k^T R_k^{-1} H_k, R_k = \Lambda_k$$

- Use the transformed measurements.

$$\bar{H}_k^T \Lambda_k^{-1} \bar{H}_k = \left[\bar{\boldsymbol{h}}_k^1 \quad \bar{\boldsymbol{h}}_k^2 \quad \dots \quad \bar{\boldsymbol{h}}_k^m\right]$$
$$\times \begin{bmatrix} \lambda_{k,1}^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_{k,2}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{k,m}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{h}_k^{1\,T} \\ \boldsymbol{h}_k^{2\,T} \\ \vdots \\ \boldsymbol{h}_k^{m\,T} \end{bmatrix}$$

$\left(P_k^+\right)^{-1} = \left(P_k^-\right)^{-1} + \sum_{i=1}^m \bar{\boldsymbol{h}}_k^i \bar{\boldsymbol{h}}_k^{i\,T} / \lambda_{k,i}, \boldsymbol{h}_k^{i\,T} = i^{th}$ row of $\bar{H}_k$

- Avoid matrix inversion with recursion.

Sequential Covariance Form

Recursion equivalent to covariance form:

For $i = 1,2, \dots, m$

$$P_k^i = \left[I_n - \boldsymbol{k}_k^i \bar{\boldsymbol{h}}_k^{i\,T}\right] P_k^{i-1}, P_k^0 = P_k^-$$
$$\boldsymbol{k}_k^i = P_k^{i-1} \bar{\boldsymbol{h}}_k^i / \left[\bar{\boldsymbol{h}}_k^{i\,T} P_k^{i-1} \bar{\boldsymbol{h}}_k^i + \lambda_{k,i}\right]$$
$$\hat{\boldsymbol{x}}_k^i = \hat{\boldsymbol{x}}_k^{i-1} + \boldsymbol{k}_k^i \left[\bar{\boldsymbol{z}}_{k,i} - \bar{\boldsymbol{h}}_k^{i\,T} \hat{\boldsymbol{x}}_k^{i-1}\right], \hat{\boldsymbol{x}}_k^0 = \hat{\boldsymbol{x}}_k^-$$
$$\hat{\boldsymbol{x}}_k^+ = \hat{\boldsymbol{x}}_k^m, \qquad P_k^+ = P_k^m$$

Proof of Covariance Formula

$$P_k^i = \left[I_n - \boldsymbol{k}_k^i \bar{\boldsymbol{h}}_k^{i\,T}\right] P_k^{i-1}$$
$$P_k^0 = P_k^-, i = 1, \dots, m$$

- Substitute for the gain

$$\boldsymbol{k}_k^i = P_k^{i-1} \bar{\boldsymbol{h}}_k^i / \left[\bar{\boldsymbol{h}}_k^{i\,T} P_k^{i-1} \bar{\boldsymbol{h}}_k^i + \lambda_{k,i}\right]$$
$$P_k^i = P_k^{i-1}$$
$$-P_k^{i-1} \bar{\boldsymbol{h}}_k^i \left(\bar{\boldsymbol{h}}_k^{i\,T} P_k^{i-1} \bar{\boldsymbol{h}}_k^i + \lambda_{k,i}\right)^{-1} \bar{\boldsymbol{h}}_k^{i\,T} P_k^{i-1}$$

Proof (Cont.)

$$P_k^i = P_k^{i-1} - P_k^{i-1}\bar{\boldsymbol{h}}_k^i\left({\bar{\boldsymbol{h}}_k^i}^T P_k^{i-1}\bar{\boldsymbol{h}}_k^i + \lambda_{k,i}\right)^{-1}{\bar{\boldsymbol{h}}_k^i}^T P_k^{i-1}$$

- Use the matrix inversion lemma

$$\left[A_1 + A_2A_4^{-1}A_3\right]^{-1}$$
$$= A_1^{-1} - A_1^{-1}A_2\left[A_4 + A_3A_1^{-1}A_2\right]^{-1}A_3A_1^{-1}$$
$$A_1 = \left(P_k^{i-1}\right)^{-1}, A_2 = \bar{\boldsymbol{h}}_k^i, \qquad A_3 = {\bar{\boldsymbol{h}}_k^i}^T, A_4 = \lambda_{k,i}$$
$$P_k^i = \left[\left(P_k^{i-1}\right)^{-1} + \bar{\boldsymbol{h}}_k^i\lambda_{k,i}^{-1}{\bar{\boldsymbol{h}}_k^i}^T\right]^{-1}, i = 1, \dots, m$$
$$\left(P_k^m\right)^{-1} = \left(P_k^-\right)^{-1} + \sum_{i=1}^{m}\bar{\boldsymbol{h}}_k^i{\boldsymbol{h}_k^i}^T/\lambda_{k,i} = \left(P_k^+\right)^{-1}$$

Innovations Process

- Zero-mean Gaussian white noise

$$\hat{\boldsymbol{z}}_k^- = H_k\hat{\boldsymbol{x}}_k^-$$
$$\tilde{\boldsymbol{z}}_k^- = \boldsymbol{z}_k - \hat{\boldsymbol{z}}_k^- = H_k\left(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k^-\right) + \boldsymbol{v}_k$$

- For an unbiased estimator $\hat{\boldsymbol{x}}_k^-$

$$E\{\tilde{\boldsymbol{z}}_k^-\} = H_kE\left\{\left(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k^-\right)\right\} + E\{\boldsymbol{v}_k\}$$
$$= H_k \times \boldsymbol{0} + \boldsymbol{0}$$
$$E\left\{\tilde{\boldsymbol{z}}_k^-{\tilde{\boldsymbol{z}}_k^-}^T\right\} = H_kP_k^-H_k^T + R_k$$
$$\tilde{\boldsymbol{z}}_k^- \sim \mathcal{N}\left(\boldsymbol{0}, H_kP_k^-H_k^T + R_k\right)$$

Innovations for Sequential Filter

- Zero−mean Gaussian white noise

$$\hat{z}_{k,i}^- = {\bar{\boldsymbol{h}}_k^i}^T\hat{\boldsymbol{x}}_k^{i-1}$$
$$\tilde{\bar{z}}_{k,i} = \bar{z}_{k,i} - \hat{z}_{k,i}^- = {\bar{\boldsymbol{h}}_k^i}^T\left(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k^{i-1}\right) + \bar{\boldsymbol{v}}_{k,i}$$

- For an unbiased estimator $\hat{\boldsymbol{x}}_k^-$

$$E\left\{\tilde{\bar{z}}_{k,i}\right\} = {\bar{\boldsymbol{h}}_k^i}^T E\left(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k^{i-1}\right) + E\left\{\bar{\boldsymbol{v}}_{k,i}\right\} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}$$
$$E\left\{\tilde{\bar{z}}_{k,i}^2\right\} = {\bar{\boldsymbol{h}}_k^i}^T P_k^{i-1}\bar{\boldsymbol{h}}_k^i + \lambda_{k,i}$$

- Proof of white uses orthogonality (skip)

Gain Formula

$$\boldsymbol{k}_k^i = P_k^{i-1}\bar{\boldsymbol{h}}_k^i / \left[{\bar{\boldsymbol{h}}_k^i}^T P_k^{i-1}\bar{\boldsymbol{h}}_k^i + \lambda_{k,i}\right]$$
$$i = 1 \dots, m$$

- Gain expression used in the covariance recursion gives the correct error covariance
- Equivalent to using the Kalman gain in the covariance filter or information filter.

Relation to Kalman Gain

$$P_k^i = \left[I_n - \boldsymbol{k}_k^i \overline{\boldsymbol{h}}_k^{i\,T}\right] P_k^{i-1}, P_k^0 = P_k^-, i = 1, \dots, m$$

$$P_k^+ = \left[I_n - \boldsymbol{k}_k^m \overline{\boldsymbol{h}}_k^{m\,T}\right] \times \cdots \times \left[I_n - \boldsymbol{k}_k^1 \overline{\boldsymbol{h}}_k^{1\,T}\right] P_k^0$$

$$= [I_n - K_k H_k] P_k^-$$

- Overall Gain

$$I_n - K_k H_k = \left[I_n - \boldsymbol{k}_k^m \overline{\boldsymbol{h}}_k^{m\,T}\right] \times \cdots \times \left[I_n - \boldsymbol{k}_k^1 \overline{\boldsymbol{h}}_k^{1\,T}\right]$$

State Estimate Recursion

$$\boldsymbol{k}_k^i = P_k^{i-1} \overline{\boldsymbol{h}}_k^i / \left[\overline{\boldsymbol{h}}_k^{i\,T} P_k^{i-1} \overline{\boldsymbol{h}}_k^i + \lambda_{k,i}\right]$$

$$I_n - K_k H_k = \left[I_n - \boldsymbol{k}_k^m \overline{\boldsymbol{h}}_k^{m\,T}\right] \times \cdots \times \left[I_n - \boldsymbol{k}_k^1 \overline{\boldsymbol{h}}_k^{1\,T}\right]$$

$$\hat{\boldsymbol{x}}_k^i = \left[I_n - \boldsymbol{k}_k^i \overline{\boldsymbol{h}}_k^{i\,T}\right] \hat{\boldsymbol{x}}_k^{i-1} + \boldsymbol{k}_k^i \bar{z}_{k,i}, \hat{\boldsymbol{x}}_k^0 = \hat{\boldsymbol{x}}_k^-,$$

$$i = 1 \dots, m$$

$$\hat{\boldsymbol{x}}_k^m = \hat{\boldsymbol{x}}_k^+ = [I_n - K_k H_k] \hat{\boldsymbol{x}}_k^- + K_k \boldsymbol{z}_k$$

- Estimate expression is correct since it gives the correct error covariance.

Example (Dan Simon)

$$x_{k+1} = 0.95 x_k + w_k$$

$$\boldsymbol{z}_k = [z_{k,1} \quad z_{k,2} \quad z_{k,3}]^T = \begin{bmatrix} 1 \\ 1/5 \\ 1/50 \end{bmatrix} x_k + \boldsymbol{v}_k$$

$$w_k \sim \mathcal{N}(0, Q), Q = 2$$

$$\boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, R), R = diag\{2,1,50\}$$

$$\hat{x}_0^+ = 1, \qquad P_0^+ = 4$$

$$\phi = 0.95, \qquad H = [1 \quad 1/5 \quad 1/50]^T$$

Covariance Filter

$$P_1^- = \phi P_0^+ \phi + Q$$

$$= (0.95)^2 \times 4 + 2 = 5.61$$

$$\hat{x}_1^- = 0.95 \hat{x}_0^+ = 0.95$$

$$K_1 = P_1^- H^T \left(H P_1^- H^T + R\right)^{-1}$$

$$= [0.6961 \quad 0.2785 \quad 6 \times 10^{-4}]$$

$$\hat{x}_1^+ = \hat{x}_1^- + K_1(\boldsymbol{z}_1 - H \hat{x}_1^-) = 5.1922$$

$$P_1^+ = (1 - K_1 H) P_1^- = 1.3923$$

Sequential Computation

- Predictor (as usual) $P_1^- = 5.61,\ \hat{x}_1^- = 0.95$
- Corrector: Measurement 1

$$P_1^0 = P_1^-, \hat{x}_1^0 = \hat{x}_1^-,\ H = [h^1 \quad h^2 \quad h^3]^T$$

$$k_1^1 = \frac{P_1^0 h^1}{h^1 P_1^0 h^1 + \lambda_1} = 0.7372$$

$$\hat{x}_1^1 = \hat{x}_1^0 + k_1^1\left(z_{1,1} - h^1 \hat{x}_1^0\right) = 4.6728$$

$$P_1^1 = \left[1 - k_1^1 h^1\right] P_1^0 = 1.4744$$

Corrector: Measurements 2,3

$$k_1^2 = \frac{P_1^1 h^2}{h^2 P_1^1 h^2 + \lambda_2} = 0.2785$$

$$\hat{x}_1^2 = \hat{x}_1^1 + k_1^2\left(z_{1,2} - h^2 \hat{x}_1^1\right) = 5.2479$$

$$P_1^2 = \left[1 - k_1^2 h^2\right] P_1^1 = 1.3923$$

$$k_1^3 = \frac{P_1^2 h^3}{h^3 P_1^2 h^3 + \lambda_3} = 6 \times 10^{-4}$$

$$\hat{x}_1^3 = \hat{x}_1^2 + k_1^3\left(z_{1,3} - h^3 \hat{x}_1^2\right) = 5.1922 = \hat{x}_1^+$$

$$P_1^3 = \left[1 - k_1^3 h^3\right] P_1^2 = 1.3923 = P_1^+$$

Conclusion

- More efficient computation for R block diagonal.
- Data transformation (see Kailath et al.): use for diagonal or constant R.

References

- R. G. Brown and P. Y. C. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, 4ed, J. Wiley, NY, 2012.
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- D. Simon, *Optimal State Estimation: Kalman, H∞, and Nonlinear Approaches*, Wiley Interscience, NY, 2006.