OUTLINE

• Errors due to linearization.
• Unscented transformation.
• Unscented Kalman filter.

PROBLEM STATEMENT

• Mean $m_x$ and covariance $P_x$ of a $n \times 1$ stochastic vector $x$.
• Find the mean $m_y$ and covariance $P_y$ of the output of a known nonlinear function $y = h(x)$.

LINEARIZATION: SCALAR EXAMPLE

• Mean $y = h(x)$
  \[ m_y = E\{h(x)\} \]
  \[ \approx E\left\{ h(m_x) + \frac{\partial h}{\partial x}\bigg|_{m_x} \tilde{x} \right\} \]
  \[ \tilde{x} = x - m_x, \quad E\{\tilde{x}\} = 0 \Rightarrow m_y \approx h(m_x) \]
• How good is the approximation $m_y = h(m_x)$?
MEAN: HIGHER ORDER TERMS

Define the operator $D_{\bar{x}} = \frac{\partial (\cdot)}{\partial x} |_{m_x} \bar{x}$, e.g. $D_{\bar{x}} h = \frac{\partial h}{\partial x} |_{m_x} \bar{x}$.

Assume: pdf of $x$ is symmetric around its mean (e.g. Gaussian) ⇒ pdf of $\bar{x}$ is symmetric around 0 (even)

$$\int_{-\infty}^{\infty} \text{odd} \times \text{even} \, d\bar{x} = 0 \Rightarrow E\{\bar{x}^i\} = 0, \text{i odd}$$

$y = h(x)$

$$m_y = h(m_x) + \frac{1}{2} E\{D_{\bar{x}}^2 h\} + \frac{1}{4} E\{D_{\bar{x}}^4 h\} + \ldots \text{(even terms)}$$

Linearization: all higher-order terms neglected.

LINEARIZATION: COVARIANCE

$$y - m_y \approx D_{\bar{x}} h = \frac{\partial h}{\partial x} |_{m_x} \bar{x} = H\bar{x}$$

$$P_y = E \{ (y - m_y)(y - m_y)^T \} = HE\{\bar{x}\bar{x}^T\}H^T$$

$$P_y = HP_xH^T$$

• Using higher-order terms, we can show that

$$P_y = HP_xH^T + E \left\{ \frac{D_{\bar{x}} h(D_{\bar{x}} h)^T}{3!} + \frac{D_{\bar{x}} h D_{\bar{x}}^2 h^T}{2!} + \frac{D_{\bar{x}}^3 h(D_{\bar{x}} h)^T}{3!} \right\} + \ldots$$

• Linearization: all higher-order terms neglected.

UNSCENTED TRANSFORMATION

• Approximate a Gaussian density with a set of deterministically chosen sample points.

• The sample points completely capture the mean and covariance of the Gaussian distribution.

• When nonlinearly transformed, the new points completely capture the mean and covariance of the new density (errors in third order and higher terms).

UNSCENTED TRANSFORMATION: PROCEDURE

• Select $N$ sigma points $\sigma_x^{(i)}, i = 0, 1, 2, \ldots, N$, with the mean $m_x$ and covariance $P_x$.

• Transform the sigma points through the known nonlinear function $\sigma_y^{(i)} = h(\sigma_x^{(i)}), i = 0, 1, 2, \ldots, N$ (projected sigma points).

• Weighted sample mean and sample covariance of the $\sigma_y^{(i)}, i = 0, 1, 2, \ldots, N$, give a good approximation of the true mean and covariance of the output $y$. 
MEAN AND COVARIANCE ESTIMATES

If the weight associated with the $i^{th}$ sigma point is denoted by $W^{(i)}$ then we have

Estimate of the output mean

$$\hat{m}_y = \sum_{i=0}^{N-1} W^{(i)} \sigma_y^{(i)}$$

Estimate of the output covariance

$$\hat{p}_y = \sum_{i=0}^{N-1} W^{(i)} \left( \sigma_y^{(i)} - \hat{m}_y \right) \left( \sigma_y^{(i)} - \hat{m}_y \right)^T$$

BASIC UT

Use the $2n$ sigma points

$$\sigma_x^{(i)} = m_x + \bar{p}_x^{(i)} \quad i = 1, ..., 2n$$

$$\bar{p}_x^{(i)} = \left( \sqrt{nP_x} \right)_T^{(i)} \quad i = 1, ..., n$$

$$\bar{p}_x^{(n+i)} = - \left( \sqrt{nP_x} \right)_T^{(i)} \quad i = 1, ..., n$$

$\left( \sqrt{nP_x} \right)_T^{(i)}$ is the $i^{th}$ row of $\sqrt{nP_x}$

$\sqrt{nP_x}$ = matrix square root of $nP_x$

$$\sqrt{nP_x} \sqrt{nP_x} = nP_x$$

$$W^{(i)} = \frac{1}{2n}, i = 1, ..., 2n$$

GENERAL UT

Uses the $2n + 1$ sigma points $\sigma_x^{(0)} = m_x$

$$\sigma_x^{(i)} = m_x + \tilde{p}_x^{(i)} \quad i = 1, ..., 2n$$

$$\tilde{p}_x^{(i)} = \left( \sqrt{(n + \kappa)P} \right)_T^{(i)} \quad i = 1, ..., n$$

$$\tilde{p}_x^{(n+i)} = - \left( \sqrt{(n + \kappa)P} \right)_T^{(i)} \quad i = 1, ..., n$$

$\kappa$ = design parameter, determines the degree of emphasis on $\sigma_x^{(0)}$ and reduces the higher-order approximation errors.

WEIGHTS FOR GENERAL UT

• Calculate mean and covariance using:

$$W^{(0)} = \frac{\kappa}{n + \kappa}$$

$$W^{(i)} = \frac{1}{2(n + \kappa)} \quad i = 1, ..., 2n$$

• For $\kappa = 0$, general UT reduces to basic UT.
OTHER UNSCENTED TRANSFORMATIONS

- Any set of sigma points whose mean and covariance are the same as those of the input signal can form an UT.
- Number of possible UTs is infinite.
- Can take higher statistical moments into account but this may require using a large number of sigma points.
- High computational load of these UTs makes them unsuitable in many practical applications.

EXERCISE: BASIC UT

- MATLAB function
- Input= (mean, covariance) of x
- Output= (mean, covariance) of y

function [my,Py]=unscented_basic(mx,Px)
    n=length(mx);
    two_n=2*n;
    nPs=chol(n*Px,'lower'); % Square root of Px

CALCULATE & MAP SAMPLE POINTS

for i=1:n % Calculate the sample points
    X{i}=mx+nPs(i,:);
    X{i+n}=mx-nPs(i,:);
end
for i=1:two_n % Map the sample points
    Y{i}=map_x(X{i}); % function map_x
end

CALCULATE MEAN

my=zeros(1,n); % Calculate mean of y
for i=1:two_n
    my=my+Y{i};
end
my=my/two_n; % Mean of y
**CALCULATE COVARIANCE**

Py=zeros(n,n); % Calculate covariance of y
for i=1:two_n
    e=Y{i}-my; % Row error vector
    Py=Py+e*e';
end
Py=Py/two_n/n; % Covariance of y

**UNSCENTED KALMAN FILTER**

- Discrete KF in which an UT is used to obtain the mean and covariance updates.
- No Jacobian computation
- Estimate the posterior expectation

\[ E\{x_k/z_k^*\} \]
\[ z_k^* = \{z_k, i = 0,1, \ldots, k\} \]

**SYSTEM MODEL**

- DT nonlinear system \( x_{k+1} = f(x_k, u_k) + w_k \)
- Nonlinear measurement model

\[ z_k = h(x_k) + v_k \]

- \( x_k = n \times 1 \) state vector
- \( z_k = m \times 1 \) measurement vector
- \( f(x_k, u_k) = \) nonlinear state transition vector.
- \( h(x_k) = \) nonlinear measurement vector

**NOISE/INITIAL ESTIMATES**

- \( w_k = \) white zero-mean process noise with covariance matrix \( Q_k \)
- \( v_k = \) white zero-mean measurement noise with covariance matrix \( R_k \).
- Uncorrelated measurement and process noise.
- Initial state \( \hat{x}_0^* \) and the initial error covariance matrix \( P_0^* \).
**CORRECTOR**

- Use \((\hat{x}_k, P_k^-)\) as \((m_x, P_x)\) in slide 10 to calculate
  \[
  \hat{x}_k^{(i)} = \hat{x}_k^- + \hat{P}_x^{(i)}, \ i = 0, 1, ..., N - 1
  \]
  \[
  \hat{z}_k^- = \sum_{i=0}^{N-1} W^{(i)} h(\hat{x}_k^{(i)})
  \]
  \[
  P_{z,k}^- = \sum_{i=0}^{N-1} \left\{ W^{(i)} \left( h(\hat{x}_k^{(i)}) - \hat{z}_k^- \right) \left( h(\hat{x}_k^{(i)}) - \hat{z}_k^- \right)^T \right\} + R_k
  \]
  \[
  P_{xz,k}^- = \sum_{i=0}^{N-1} \left\{ W^{(i)} \left( \hat{x}_k^{(i)} - \hat{x}_k^- \right) \left( h(\hat{x}_k^{(i)}) - \hat{z}_k^- \right)^T \right\}
  \]

**KF: CORRECTOR**

Linear measurement eqn. \(z_k = H_k x_k + v_k\)

\[
\hat{z}_k^- = H_k \hat{x}_k^- 
\]

\[
P_{z,k}^- = P_k^- H_k^T, \quad P_{xz,k}^- = H_k P_k^- H_k^T + R_k
\]

use the discrete KF equations

\[
K_k = P_{xz,k}^- (P_{z,k}^-)^{-1} = P_k^- H_k^T \left( H_k P_k^- H_k^T + R_k \right)^{-1}
\]

\[
\hat{x}_k^+ = \hat{x}_k^- + K_k (z_k - H_k \hat{x}_k^-)
\]

\[
P_k^+ = (I - K_k H_k) P_k^-
\]

\[
K_k = \text{Kalman gain.}
\]

**CORRECTOR**

Use expressions obtained earlier \(K_k = C_{xz} C_z^{-1}\)

\[
P_k^+ = C_{x|z} = C_x - C_{xz} C_z^{-1} C_{xz} = C_x - K_k C_z K_k^T
\]

Gain: \(K_k = P_{xz,k}^- (P_{z,k}^-)^{-1}\)

A Posteriori State Estimate

\[
\hat{x}_k^+ = \hat{x}_k^- + K_k (z_k - \hat{z}_k^-)
\]

A Posteriori Error Covariance Matrix

\[
P_k^+ = P_k^- - K_k P_{z,k}^- K_k^T
\]

**KF: PREDICTOR**

- Use the UT with \((\hat{x}_k^+, P_k^+)\) as \((m_x, P_x)\) in slide 10 to generate \(N\) points \(\hat{x}_k^{(i)} = \hat{x}_k^+ + \hat{P}_x^{(i)}, \ i = 0, 1, ..., N - 1\)

- Obtain the mean \(\hat{x}_{k+1}^-\) and covariance \(P_{k+1}^-\) using the nonlinear transformation

\[
\hat{x}_{k+1}^- = f(\hat{x}_k, u_k), \ i = 0, 1, ..., N - 1
\]

\[
\hat{x}_{k+1}^- = \sum_{i=0}^{N-1} W^{(i)} \hat{x}_k^{(i)} + W^{(i)}, \ i = 0, 1, ..., N - 1
\]

\[
P_k^- = \sum_{i=0}^{N-1} \left\{ W^{(i)} (\hat{x}_k^{(i)} - \hat{x}_{k+1}^-) (\hat{x}_k^{(i)} - \hat{x}_{k+1}^-)^T \right\} + Q_k
\]
EXAMPLE: FALLING BODY

- Range measuring device at altitude $a$ at a horizontal distance $M$ from the body.

\[
\begin{aligned}
\dot{x}_1 &= x_2 + w_1 \\
\dot{x}_2 &= \rho_0 \exp \left( -\frac{x_1}{2K} \right) x_2^2 x_3 - g + w_2 \\
\dot{x}_3 &= w_3 \\
z(t_k) &= \sqrt{M^2 + (x_1(t_k) - a)^2} + v_k
\end{aligned}
\]

$\rho_0 =$ air density at sea level
$K =$ constant relating air density and altitude

MODEL

\[
\begin{aligned}
f(x_k, u_k) &= \begin{bmatrix} x_2 \\ \rho_0 \exp \left( -\frac{x_1}{2K} \right) x_2^2 x_3 - g \end{bmatrix} \\
h(x_k) &= \sqrt{M^2 + (x_1(t_k) - a)^2} \\
w_k &= [w_{1k} \ w_{2k} \ w_{3k}]^T \\
Q_k &= [0]
\end{aligned}
\]

- May be better to add small noise terms.

EXTENDED KF

\[
\dot{x}_2 = \rho_0 \exp \left( -\frac{x_1}{K} \right) x_2^2 x_3 - g + w_2
\]

- Use Maple to linearize
  with(VectorCalculus)
  Jacobian([x1dot,x2dot,x3dot],[x1,x2,x3])
  diff(h,x1)

- Use Van Loan approach to discretize.
- Use the forward Euler approximation for the predictor: $\dot{x}_2 \approx (x_2(t_{k+1}) - x_2(t_k))/\Delta t$

MAPLE RESULTS

\[
\frac{\partial f}{\partial x} =
\begin{bmatrix}
0 & \frac{1}{x_1} & 0 \\
-\frac{\rho_0}{K} \exp \left( -\frac{x_1}{K} \right) x_2^2 x_3 & 2\rho_0 \exp \left( -\frac{x_1}{K} \right) x_2 x_3 & \rho_0 \exp \left( -\frac{x_1}{K} \right) x_2^2 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[
H_k = \frac{\partial h}{\partial x}\bigg|_{x_1(t_k)} = \begin{bmatrix}
\frac{x_1(t_k) - a}{\sqrt{M^2 + (x_1(t_k) - a)^2}} & 0 & 0 \\
\end{bmatrix}
\]
PARAMETER VALUES

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_0$</td>
<td>2</td>
<td>lb. sec$^2$/ft$^4$</td>
</tr>
<tr>
<td>$g$</td>
<td>32.2</td>
<td>ft./sec</td>
</tr>
<tr>
<td>$K$</td>
<td>20,000</td>
<td>ft.</td>
</tr>
<tr>
<td>$M$</td>
<td>100,000</td>
<td>ft.</td>
</tr>
<tr>
<td>$a$</td>
<td>100,000</td>
<td>ft.</td>
</tr>
<tr>
<td>$E{w_i^2}, i = 1,2,3$</td>
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<td></td>
</tr>
<tr>
<td>$E{v_k^2}$</td>
<td>10,000</td>
<td>ft.$^2$</td>
</tr>
</tbody>
</table>

INITIAL CONDITIONS

$x_0 = [300,000 \quad -20,000 \quad 0.001]^T$

$x_0^+ = x_0$

$P_0^+ = diag\{1 \times 10^6, \ 4 \times 10^6, \ 10\}$

- Simulate the system, the extended KF and UKF.
- UKF gives a smaller estimation error.