Discrete-Time State-Space Equations

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Outline

- Discrete-time (DT) state equation from solution of continuous-time state equation.
- Expressions in terms of constituent matrices.
- Solution of DT state equation.
- Example.

Solution of State Equation

$$\boldsymbol{x}(t_f) = e^{A(t_f - t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t_f} e^{A(t_f - \tau)} B\boldsymbol{u}(\tau)d\tau$$

- Obtain difference equation from the solution of the analog state, over a sampling period T.
- Initial time $t_0 = kT$, and final time $t_f = (k+1)T$

$$\boldsymbol{x}(k+1) = e^{AT}\boldsymbol{x}(k) + \int_{kT}^{(k+1)T} e^{A((k+1)T - \tau)} B\boldsymbol{u}(\tau)d\tau$$

$\boldsymbol{x}(k) = \boldsymbol{x}(kT)$= state vector at time kT

Digital Control

- Piecewise constant inputs over a sampling period.

$$\boldsymbol{u}(t) = \boldsymbol{u}(k), \qquad kT \le t < (k+1)T$$

- Move input outside the integral

$$\boldsymbol{x}(k+1) = e^{AT}\boldsymbol{x}(k) + \left[\int_{kT}^{(k+1)T} e^{A\left((k+1)T-\tau\right)} B\, d\tau\right] \boldsymbol{u}(k)$$

Simplify Integral

- Change of variable of integration

$$\lambda = (k+1)T - \tau, \lambda = \begin{cases} 0, & \tau = (k+1)T \\ T, & \tau = kT \end{cases}$$

$$d\lambda = -d\tau$$

$$\int_{kT}^{(k+1)T} e^{A\left((k+1)T-\tau\right)} B d\tau = \int_{T}^{0} e^{A\lambda} B(-d\lambda)$$

$$= \int_{0}^{T} e^{A\lambda} B d\lambda$$

Discrete State-Space Equations

$$\boldsymbol{x}(k+1) = A_d\boldsymbol{x}(k) + B_d\boldsymbol{u}(k), k = 0,1, \ldots$$

$$A_d = e^{AT}, B_d = \int_0^T e^{A\tau} B d\tau$$

$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$

- Obtain A_d using state-transition matrix of A
- Integrate to obtain B_d

State & Input Matrices

$A_d = n \times n$ discrete state matrix

$B_d = n \times m$ discrete input matrix

(same orders as their continuous counterparts).

- Discrete state matrix = state transition matrix of the analog system evaluated at the sampling period T.
- Properties of the matrix exponential: integral of the matrix exponential for invertible matrix A

$$\int e^{At} dt = A^{-1} e^{At} = e^{At} A^{-1}$$

$$B_d = A^{-1}[e^{AT} - I_n]B = [e^{AT} - I_n]A^{-1}B$$

Constituent Matrices

- Use expansion of the matrix exponential in terms of the constituent matrices.
- Eigenvalues of discrete state matrix related to those of the analog system.

$$A_d = \sum_{i=1}^{n} Z_i e^{\lambda_i t} \Big]_{t=T} = \sum_{i=1}^{n} Z_i e^{\lambda_i T}$$

Input Matrix

$$B_d = \int_0^T e^{A\tau} B d\tau = \int_0^T \left(\sum_{i=1}^{n} Z_i e^{\lambda_i \tau} \right) B d\tau$$

$$B_d = \sum_{i=1}^{n} Z_i B \int_0^T e^{\lambda_i \tau} d\tau$$

- Scalar integrands: easily evaluate integral.

$$B_d = \begin{cases} \sum_{i=1}^{n} Z_i B \left[\dfrac{1 - e^{\lambda_i T}}{-\lambda_i} \right], & \lambda_i \neq 0 \\ Z_1 B T + \sum_{i=2}^{n} Z_i B \left[\dfrac{1 - e^{\lambda_i T}}{-\lambda_i} \right], & \lambda_1 = 0 \end{cases}$$

- Assume distinct eigenvalues (only one zero eigenvalue)

Discrete-time State-space Representation

- Discrete state & output equation.
- Discrete-time state equation: approximately valid for a general input vector $\boldsymbol{u}(t)$ provided that the sampling period T is sufficiently short.

$$\boldsymbol{x}(k+1) = A_d\boldsymbol{x}(k) + B_d\boldsymbol{u}(k)$$

- Output equation evaluated at time kT

$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$

Example 7.15

- Obtain the DT state equations for the system of Example 7.7

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

- for a sampling period $T = 0.01$ s.
- Solution: From Example 7.7, the state-transition matrix is

$$e^{At} = \begin{bmatrix} 10 & 11 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{e^{0}}{10} + \begin{bmatrix} 0 & -10 & -1 \\ 0 & 10 & 1 \\ 0 & -10 & -1 \end{bmatrix} \frac{e^{-t}}{9} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & -10 & -10 \\ 0 & 100 & 100 \end{bmatrix} \frac{e^{-10t}}{90}$$

Discrete state matrix

$$A_d = e^{A\times 0.01}$$
$$= \begin{bmatrix} 10 & 11 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{e^{0}}{10} + \begin{bmatrix} 0 & -10 & -1 \\ 0 & 10 & 1 \\ 0 & -10 & -1 \end{bmatrix} \frac{e^{-0.01}}{9}$$
$$+ \begin{bmatrix} 0 & 1 & 1 \\ 0 & -10 & -10 \\ 0 & 100 & 100 \end{bmatrix} \frac{e^{-10\times 0.01}}{90}$$

- Simplifies to

$$A_d = \begin{bmatrix} 1.0 & 0.1 & 0.0 \\ 0 & 0.9995 & 0.0095 \\ 0 & 0.0947 & 0.8954 \end{bmatrix}$$

Discrete-time Input matrix

$$B_d = Z_1 B(0.01) + Z_2 B[1 - e^{-0.01}] + Z_3 B\left[\frac{1 - e^{-10\times 0.01}}{10}\right]$$

$$= \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix} + \frac{10}{9}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}[1 - e^{-0.01}] + \frac{1}{9}\begin{bmatrix} 1 \\ -10 \\ 100 \end{bmatrix}\left[\frac{1 - e^{-10\times 0.01}}{10}\right]$$

- Simplifies to $B_d = \begin{bmatrix} 1.622 \times 10^{-6} \\ 4.821 \times 10^{-4} \\ 9.468 \times 10^{-2} \end{bmatrix}$

MATLAB

Obtain (A_d, B_d, C, D) from (A, B, C, D)

```
» A=[1,2;3,4]; B=[0;1];C=[1,1]; D=0;
» p=ss(A, B, C, D); % State-space matrices
» pd = c2d(p,0.01)
% piecewise constant input (default)
```

Alternatively, use the MATLAB commands

```
» ad = expm(a * 0.05)
» bd = a\ (ad-eye(3) )* b
```

MATLAB Output

```
a =
            x1        x2
   x1      1.01  0.02051
   x2  0.03076     1.041
b =
              u1
   x1  0.0001017
   x2     0.0102
c =
       x1  x2
   y1   1   1
d =
       u1
   y1   0
```

Sampling time: 0.01

- Discrete-time model.

Solution of DT State-Space Equation

- DT State Equation: state at time k in terms of the initial condition vector $\boldsymbol{x}(0)$ and the input sequence $\boldsymbol{u}(k), k = 0, 1, \ldots, k-1$.

$$\boldsymbol{x}(k+1) = A_d\boldsymbol{x}(k) + B_d\boldsymbol{u}(k)$$

- At $k = 0, 1,$ we have

$$\boldsymbol{x}(1) = A_d\boldsymbol{x}(0) + B_d\boldsymbol{u}(0)$$
$$\boldsymbol{x}(2) = A_d\boldsymbol{x}(1) + B_d\boldsymbol{u}(1)$$
$$= A_d^2\boldsymbol{x}(0) + A_dB_d\boldsymbol{u}(0) + B_d\boldsymbol{u}(1)$$

Solution by Induction

$$\boldsymbol{x}(2) = A_d^2 \boldsymbol{x}(0) + \sum_{i=0}^{2-1} A_d^{2-i-1} B_d \boldsymbol{u}(i)$$

$$\boldsymbol{x}(k) = A_d^k \boldsymbol{x}(0) + \sum_{i=0}^{k-1} A_d^{k-i-1} B_d \boldsymbol{u}(i)$$

$$\boldsymbol{x}(k) = A_d^{k-k_0} \boldsymbol{x}(k_0) + \sum_{i=k_0}^{k-1} A_d^{k-i-1} B_d \boldsymbol{u}(i)$$

- State-transition matrix for the DT system A_d^k.
- State-transition matrix for time-varying DT system:
  - not a matrix power
  - dependent on both time k and initial time k_0 .
- Solution=zero-input response+ zero-state response

Output Solution

- Substitute in output equation

$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$

$$= C\left\{A_d^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1} A_d^{k-i-1}B_d\boldsymbol{u}(i)\right\} + D\boldsymbol{u}(k)$$

Z-Transform Solution of DT State Equation

- z-transform the discrete-time state equation

$$z\boldsymbol{X}(z) - z\boldsymbol{x}(0) = A_d\boldsymbol{X}(z) + B_d\boldsymbol{U}(z)$$

$$\boldsymbol{X}(z) = [zI_n - A_d]^{-1}\{z\boldsymbol{x}(0) + B_d\boldsymbol{U}(z)\}$$

$$[zI_n - A_d]^{-1}z = \left[I_n - \frac{1}{z}A_d\right]^{-1}$$

$$= I_n + A_d z^{-1} + A_d^2 z^{-2} + \cdots + A_d^i z^{-i} + \cdots$$

Inverse z-transform

- Inverse z-transform $[zI_n - A_d]^{-1}z$

$$[zI_n - A_d]^{-1}z = I_n + A_d z^{-1} + \cdots + A_d^i z^{-i} + \cdots$$

$$\mathcal{Z}^{-1}\{[zI_n - A_d]^{-1}z\} = \left\{I_n, A_d, A_d^2, \ldots, A_d^i, \ldots\right\}$$

$$= \left\{A_d^k\right\}_{k=0}^{\infty}$$

- Analogous to the scalar transform pair

$$\frac{z}{z - a_d} \overset{\mathcal{Z}}{\leftrightarrow} \left\{a_d^k\right\}_{k=0}^{\infty}$$

Matrix Inversion

- Evaluate using the Leverrier algorithm.
- Partial fraction expansion then multiply by z.

$$[zI - A_d]^{-1} z = z \frac{adj[zI - A_d]}{\det[zI - A_d]}$$

$$= \frac{P_0 z + P_1 z^2 + \cdots + P_{n-1} z^n}{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n}$$

$$= \sum_{i=1}^{n} \frac{z}{z - \lambda_i} Z_i \,, \qquad A_d^k = \sum_{i=1}^{n} Z_i \lambda_i^k$$

DT State Matrix

$$A_d = \sum_{i=1}^{n} Z_i \lambda_i = \sum_{i=1}^{n} Z_i(A) e^{\lambda_i(A)T}$$

- Parentheses: pertaining to the CT state matrix A.
- Equality for any sampling period T and any matrix A

$$Z_i = Z_i(A)$$
$$\lambda_i = e^{\lambda_i(A)T}$$

- Same constituent matrices for DT state matrix & CT state matrix A
- DT eigenvalues are exponential functions of the CT eigenvalues times the sampling period.

Zero-state Response

$$\boldsymbol{x}_{ZS}(t) = \{[zI_n - A_d]^{-1}z\}z^{-1}B_d\boldsymbol{U}(z)$$

- Known inverse transform for $\{.\}$ term.
- Multiplication by z^{-1}: delay by T.

Convolution theorem: inverse of product is the convolution summation

$$\boldsymbol{x}_{zs}(k) = \sum_{i=0}^{k-1} A_d^{k-i-1} B_d \boldsymbol{u}(i), \qquad A_d^k = \sum_{j=1}^{n} Z_j e^{\lambda_j(A)kT}$$

$$\boldsymbol{x}_{zs}(k) = \sum_{i=0}^{k-1} \sum_{j=1}^{n} Z_j e^{\lambda_j(A)[k-i-1]T} B_d \boldsymbol{u}(i)$$

Alternative Expression

$$\boldsymbol{x}_{zs}(k) = \sum_{i=0}^{k-1} \sum_{j=1}^{n} Z_j e^{\lambda_j(A)[k-i-1]T} B_d \boldsymbol{u}(i)$$

- Change the order of summation

$$\boldsymbol{x}_{zs}(k) = \sum_{j=1}^{n} Z_j e^{\lambda_j(A)[k-1]T} \left[\sum_{i=0}^{k-1} B_d \boldsymbol{u}(i) e^{-\lambda_j(A)iT} \right]$$

- Useful when the i summation has a closed form.

Example 7.16

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(a) Solve the state equation for a unit step input and the initial condition vector $\boldsymbol{x}(0) = [1 \quad 0]^T$

(b) Use the solution to obtain the discrete-time state equations for a sampling period of 0.1s.

(c) Solve the discrete-time state equations with the same initial conditions and input as in (a) and verify that the solution is the same as that of (a) evaluated at multiples of the sampling period T.

Solution

(a) The resolvent matrix

$$\Phi(s) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2 + 3s + 2}$$

$$= \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}}{(s+1)(s+2)}$$

- Partial fraction expansions

$$\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \frac{1}{(s+1)(s+2)} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \left(\frac{1}{s+1} + \frac{-1}{s+2} \right)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{s}{(s+1)(s+2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{-1}{s+1} + \frac{2}{s+2} \right)$$

State-transition Matrix

$$\Phi(s) = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}}{(s+1)(s+2)}$$

$$= \frac{-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}}{(s+1)} + \frac{2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}}{(s+2)}$$

$$= \frac{\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}}{(s+1)} + \frac{\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}{(s+2)}$$

$$\phi(t) = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} e^{-2t}$$

Zero-input Response

$$\boldsymbol{x}_{ZI}(t) = e^{At}\boldsymbol{x}(0)$$

$$= \left\{ \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} e^{-2t} \right\} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$$

Zero-state Response

$$\boldsymbol{x}_{ZS}(t)$$

$$= \left\{\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} e^{-2t}\right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix} * 1(t)$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} * 1(t) + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t} * 1(t)$$

$$\boldsymbol{x}_{ZS}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1 - e^{-t}) + \begin{bmatrix} -1 \\ 2 \end{bmatrix} (1 - e^{-2t})/2$$

$$= \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} - \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-2t}$$

Total Response

$$\boldsymbol{x}(t) = \boldsymbol{x}_{ZI}(t) + \boldsymbol{x}_{ZS}(t)$$

$$\boldsymbol{x}(t) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$$

$$+ \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} - \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-2t}$$

- At the sampling points:

t = multiples of 0.1s

(b) Discrete-time state equations

$$A_d = \phi(0.1) = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} e^{-0.1} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} e^{-0.2}$$

$$= \begin{bmatrix} 0.9909 & 0.0861 \\ -0.1722 & 0.7326 \end{bmatrix}$$

$$B_d = \left\{ \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} (1 - e^{-0.1}) + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \frac{(1 - e^{-0.2})}{2} \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.1} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{e^{-0.2}}{2} = \begin{bmatrix} 0.0045 \\ 0.0861 \end{bmatrix}$$

$$\boldsymbol{x}(1) = A_d \boldsymbol{x}(0) + B_d u(0)$$

CT system response to a step input of duration one sampling period=response of CT system due to a piecewise constant input over one period

Zero-input Response

$$A_d^k = \Phi(0.1k) = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} e^{-0.1k} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} e^{-0.2k}$$

$$\boldsymbol{x}_{ZI}(k) = \Phi(0.1k)\boldsymbol{x}(0)$$

$$= \left\{ \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} e^{-0.1k} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} e^{-0.2k} \right\} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{x}_{ZI}(k) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-0.1k} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-0.2k}$$

- Same as the zero-input response of the continuous-time system

$$\boldsymbol{x}_{ZI}(t) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$$

at all sampling points $k = 0, 1, 2, \ldots$

z-transform of the zero-state response

$$\Phi(z) = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \frac{z}{z - e^{-0.1}} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \frac{z}{z - e^{-0.2}}$$

$$\boldsymbol{X}_{ZS}(z) = \Phi(z) z^{-1} B_d \, U(z)$$

$$= \left\{ \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \frac{z}{z-e^{-0.1}} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \frac{z}{z-e^{-0.2}} \right\} \begin{bmatrix} 0.0045 \\ 0.861 \end{bmatrix} \frac{z^{-1} \, z}{z-1}$$

$$= 9.5163 \times 10^{-2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{z}{(z - e^{-0.1})(z - 1)}$$

$$+ 9.0635 \times 10^{-2} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \frac{z}{(z - e^{-0.2})(z - 1)}$$

Partial Fractions

$$\frac{z}{(z-e^{-0.1})(z-1)} = \frac{1}{1-e^{-0.1}}\left[\frac{z}{z-1} + \frac{(-1)z}{z-e^{-0.1}}\right]$$

$$= 10.5083\left[\frac{z}{z-1} + \frac{(-1)z}{z-0.9048}\right]$$

$$\frac{z}{(z-e^{-0.2})(z-1)} = \frac{1}{1-e^{-0.2}}\left[\frac{z}{z-1} + \frac{(-1)z}{z-e^{-0.1}}\right]$$

$$= 5.5167\left[\frac{z}{z-1} + \frac{(-1)z}{z-0.8187}\right]$$

Expand zero-state response

$$\boldsymbol{X}_{ZS}(z)$$
$$= \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \frac{z}{z-1} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{z}{z-e^{-0.1}} - \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \frac{z}{z-e^{-0.2}}$$

$$\boldsymbol{x}_{ZS}(k) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.1k} - \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-0.2k}$$

Same as zero-state response for the CT system

$$\boldsymbol{x}_{ZS}(t) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} - \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-2t}$$

at time $t = 0.1\,k, k \ = \ 0,1,2, \dots$

Zero-state response

$$\boldsymbol{x}_{zs}(k) = \sum_{i=0}^{k-1}\sum_{j=1}^{n} Z_j e^{\lambda_j(A)[k-i-1]T} B_d \boldsymbol{u}(i)$$

$$\boldsymbol{x}_{zs}(k) = \sum_{j=1}^{n} Z_j B_d e^{\lambda_j(A)[k-1]T}\left[\sum_{i=0}^{k-1}\boldsymbol{u}(i)e^{-\lambda_j(A)iT}\right]$$

$$\boldsymbol{u}(k) = \mathbf{1} = [1 \quad \dots \quad 1]^T, \qquad \sum_{i=1}^{k-1} a^i = \frac{1-a^k}{1-a}, \qquad a \neq 1$$

$$\boldsymbol{x}_{zs}(k) = \sum_{j=1}^{n} Z_j B_d \mathbf{1} e^{\lambda_j(A)[k-1]T}\left[\frac{1-e^{\lambda_j(A)kT}}{1-e^{\lambda_j(A)T}}\right]$$

z-Transfer Function

$$\boldsymbol{x}(k+1) = A_d\boldsymbol{x}(k) + B_d\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k)$$
$$\boldsymbol{Y}(z) = C\boldsymbol{X}(s) + D\boldsymbol{U}(z)$$

- For zero initial conditions

$$z\boldsymbol{X}(z) = A_d\boldsymbol{X}(z) + B_d\boldsymbol{U}(z)$$

$$\boldsymbol{X}(z) = (zI_n - A_d)^{-1}B_d\boldsymbol{U}(z)$$

- Substitute

$$\boldsymbol{Y}(z) = C(zI_n - A_d)^{-1}B_d\boldsymbol{U}(z) + D\boldsymbol{U}(z)$$
$$= \{C(zI_n - A_d)^{-1}B_d + D\}U(z) = G(z)U(z)$$

Impulse Response & Modes

$$G(z) = C(zI_n - A_d)^{-1}B_d + D$$

- Inverse transform of the transfer function

$$G(z) = C(zI_n - A_d)^{-1}B_d + D \overset{\mathcal{Z}}{\leftrightarrow} G(k) = \begin{cases} CA_d^{k-1}B_d, & k \geq 1 \\ D, & k = 0 \end{cases}$$

- Substitute in terms of constituent matrices

$$G(z) = \sum_{i=1}^{n} CZ_iB_d \frac{1}{z - \lambda_i} + D \overset{\mathcal{Z}}{\leftrightarrow} G(k) = \begin{cases} \sum_{i=1}^{n} CZ_iB_d\lambda_i^{k-1}, & k \geq 1 \\ D, & k = 0 \end{cases}$$

Poles and Stability

- poles= eigenvalues of discrete-time state matrix $A_d =$ exponential functions of $\lambda_i(A)$ (continuous-time state matrix A).
- For stable $A, \lambda_i(A)$ have negative real parts and have magnitude less than unity.
- Discretization yields a stable DT system for a stable CT system.

Minimal Realizations

- Product CZ_jB can vanish & eliminate eigenvalues from the transfer function: if $CZ_j = \mathbf{0}, Z_jB = \mathbf{0}$, or both.
- If cancellation occurs, the system is said to have an
  - $CZ_j = \mathbf{0}$ **:** output-decoupling zero at λ_j
  - $Z_jB = \mathbf{0}$**:** an input-decoupling zero at λ_j
  - $CZ_j = \mathbf{0}, \& \, Z_jB = \mathbf{0}$**:** an input-output-decoupling zero at λ_j
- Poles of the reduced transfer function are a subset of the eigenvalues of the state matrix A_d.
- A state-space realization that leads to *pole*-zero cancellation is said to be reducible or nonminimal.
- If no cancellation occurs, the realization is said to be irreducible or minimal.

Decoupling Modes

- Output-decoupling zero at λ_j: the forced system response does not include the mode λ_j^k.
- Input-decoupling zero: the mode is decoupled from or unaffected by the input.
- Input-output-decoupling zero: the mode is decoupled both from the input and the output.
- These properties are related to the concepts of controllability and observability discussed later in this chapter.

Example 7.17

- Obtain the z-transfer function for the position control system of Example 7.16

(a) With x_1 as output. (b) With $x_1 + x_2$ as output.

The resolvent matrix and input matrix were obtained in Example 7.16.

Solution

$$\Phi(z) = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \frac{z}{z - e^{-0.1}} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \frac{z}{z - e^{-0.2}}$$

$$B = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.1} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \frac{e^{-0.2}}{2} = \begin{bmatrix} 0.0045 \\ 0.0861 \end{bmatrix}$$

(a)Output x_1: $C = [1 \quad 0],\ \ D = 0$

$$G(z) = [1 \quad 0] \left\{ \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \frac{1}{z - e^{-0.1}} + \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \frac{1}{z - e^{-0.2}} \right\} \begin{bmatrix} 0.0045 \\ 0.0861 \end{bmatrix}$$

$$= \frac{9.5163 \times 10^{-2}}{z - e^{-0.1}} - \frac{9.0635 \times 10^{-2}}{z - e^{-0.2}}$$

(c) Output x_1+x_2

$$G(z) = \begin{bmatrix}1 & 1\end{bmatrix}\left\{\begin{bmatrix}2 & 1\\ -2 & -1\end{bmatrix}\frac{1}{z-e^{-0.1}} + \begin{bmatrix}-1 & -1\\ 2 & 2\end{bmatrix}\frac{1}{z-e^{-0.2}}\right\}\begin{bmatrix}0.0045\\ 0.0861\end{bmatrix}$$

$$= \frac{0}{z-e^{-0.1}} + \frac{9.0635\times 10^{-2}}{z-e^{-0.2}}$$

1. Output-decoupling zero at $e^{-0.1}$ since $CZ_1 = [0]$
2. System response to any input does not include the decoupling term.

Step Response

$$Y(z) = \frac{9.0635 \times 10^{-2}}{z - e^{-0.2}} \times \frac{z}{z-1}$$

$$= \frac{9.0635 \times 10^{-2}}{1 - e^{-0.2}} \left[\frac{z}{z-1} - \frac{z}{z - e^{-0.2}}\right]$$

$$= 0.5 \left[\frac{z}{z-1} - \frac{z}{z - e^{-0.2}}\right]$$

- Z-transform inverse

$$y(k) = 0.5[1 - e^{-0.2k}]$$

z-Transfer Function: MATLAB

- Let $T = 0.05s$

» **P = ss(Ad, Bd, C, D, 0.05)**

» **g = tf(P)** % Obtain z-domain transfer function

» **zpk(g)** % Obtain transfer function poles and zeros

- The command reveals that the system has a zero at 0.9048 and poles at (0.9048, 0.8187) with a gain of 0.09035.
- With pole-zero cancellation, the transfer function is the same as that of Example 7.17(b).

» **minreal(g)** % Cancel poles and zeros