Difference Equations

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Outline

• Difference equations as models of physical systems: example.
• Types of difference equations
  – Linear.
  – Nonlinear.
DT Model for Digital Control

• Analog plant
• Piecewise constant input updated periodically.
• Good approximations that allow us to obtain a simple model.
• Model as a difference equation.

\[ T_{k-1} T_k T_{k+1} \]

Applying the control signal: $T$, and the state of the system: $(k-1)T, kT, (k+1)T$.
Example: Tank Control System

• Adjust input flow rate to maintain a constant fluid level.
• Mathematically model the tank.
• Obtain a discrete-time model for the system with piecewise constant inflow $q_i$. 
Example: Solution

- Fluid outflow and level nonlinearly related.
- Use a linear model if fluid level varies around a constant value.

\[ h = \text{perturbation in tank level from nominal}. \]
\[ q_0 = \text{perturbation in the outflow from the tank from a nominal level } Q. \]
\[ R = \text{fluid resistance of the valve}. \]
Volumetric Balance

rate of fluid volume increase

\[ \frac{dC(H + h)}{dt} = (q_i + Q) - (q_o + Q) \]

\( C = \) area of the tank = fluid capacitance.
\( \tau = RC = \) fluid time constant for the tank.
\( H(Q) = \) nominal height (flow rate)

- Substitute from linearized valve equation

\[ \frac{dh}{dt} = -\frac{h}{\tau} + \frac{q_i}{C} \]
Differential Equation Solution

\[ h(t) = e^{-(t-t_0)/\tau} h(t_0) + \frac{1}{C} \int_{t_0}^{t} e^{-(t-\lambda)T/\tau} q_i(\lambda) d\lambda \]

\[ q_i(t) = q_i(k) = constant, \; t \in [kT, (k + 1)T) \]

- Solution over any sampling period:
  \[ t_0 = kT, \quad t = (k + 1)T \]

- **Note:** \( h(k) \) means \( h(kT) \)

\[ h(k + 1) = e^{-T/\tau} h(k) + R[1 - e^{-T/\tau}] q_i(k) \]
Remarks

• The discrete-time model obtained in the Example is known as a difference equation.

• For a linear time-invariant analog plant, we have a linear time-invariant difference equation.

• Example

\[ h(k + 1) = e^{-T/\tau}h(k) + R[1 - e^{-T/\tau}]q_i(k) \]
Difference Equations

- Nonlinear Difference Equation

\[ y(k + n) = f \left[ y(k + n - 1), y(k + n - 2), \ldots, y(k + 1), y(k), u(k + n), u(k + n - 1), \ldots, u(k + 1), u(k) \right] \]

- Linear Difference Equation

\[
\begin{align*}
y(k + n) &+ a_{n-1}y(k + n - 1) + a_{n-2}y(k + n - 2) + \cdots \\
&+ a_1y(k + 1) + a_0y(k) \\
&= b_nu(k + n) + b_{n-1}u(k + n - 1) + \cdots + b_1u(k + 1) + b_0u(k)
\end{align*}
\]
Determine the order of the equation. Is the equation (a) linear? (b) time-invariant? (c) homogeneous?

i. $y(k + 3) + 0.2y(k + 1) + 0.01y(k) = 0$

ii. $y(k + 2) + e^{-0.2k}y(k + 1) + 0.1y(k) = u(k)$

iii. $y(k + 5) + y(k + 1) + 0.1y^3(k) = 0.1u(k)$
Example (i)

\[ y(k + 3) + 0.2y(k + 1) + 0.01y(k) = 0 \]

Order, linear, time-invariant, homogeneous?

- **Third order.**
- All terms linear and have constant coefficients \( \Rightarrow \text{LTI}. \)
- No forcing function appears in the equation \( \Rightarrow \text{homogeneous}. \)
Example (ii)

- $y(k + 2) + e^{-0.2k}y(k + 1) + 0.1y(k) = u(k)$
- Order, linear, time-invariant, homogeneous?
- **Second order.**
- Second coefficient is time-dependent but all the terms are linear $\Rightarrow$ **linear time varying.**
- Forcing function $\Rightarrow$ **nonhomogeneous.**
Example (iii)

\[ y(k + 5) + y(k + 1) + 0.1y^3(k) = 0.1u(k) \]

Order, linear, time-invariant, homogeneous?

- **Fifth order.**
- Nonlinear function of \( y(k) \) ⇒ **nonlinear.**
- Forcing function ⇒ **nonhomogeneous.**
- No terms depending explicitly on time ⇒ **time invariant.**