Finite Settling Time Design

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Outline

• Control system synthesis.
• Finite settling for DT systems.
• Finite settling time controllers.
• Deadbeat controllers.
• Example.
• Intersample behavior.
Control System Synthesis

- Desired closed loop transfer function $G_{cl}(z)$ is known.
- Select a controller $C(z)$ to achieve the desired closed-loop transfer function.
Solve for the Controller

\[ G_{cl}(z) = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \]

\[ C(z) = \frac{1}{G_{ZAS}(z)} \left[ \frac{G_{cl}(z)}{1 - G_{cl}(z)} \right] \]

\( C(z) \) must be realizable and must yield an asymptotically stable system.
Causality

\[
G_{ZAS}(z) = \frac{N_{ZAS}(z)}{D_{ZAS}(z)} = \frac{N_{ol}(z)}{D_{ZAS}(z)} z^{-l}
\]

\[
G_{cl}(z) = \frac{N_{cl}(z)}{D_{cl}(z)} = \frac{N_{cl1}(z)}{D_{cl}(z)} z^{-l}
\]

\[
C(z) = \frac{1}{G_{ZAS}(z)} \left[ \frac{G_{cl}(z)}{1 - G_{cl}(z)} \right]
\]

\[
= \frac{1}{N_{ZAS}(z)} \left[ \frac{N_{cl}(z)}{D_{cl}(z) - N_{cl}(z)} \right]
\]

\[
= \frac{1}{N_{ol}(z) z^{-l}} \left[ \frac{N_{cl1}(z) z^{-l}}{D_{cl}(z) - N_{cl1}(z) z^{-l}} \right]
\]
Causality Conditions

• Causal controller must have
  – More poles than zeros, or as many poles as zeros.
  – No time advance.
First Causality Condition

\[ C(z) = \frac{D_{ZAS}(z)}{N_{ZAS}(z)} \left[ \frac{N_{cl}(z)}{D_{cl}(z) - N_{cl}(z)} \right] \]

• Causal controller must have more poles than zeros, or as many poles as zeros.
• Closed-loop and plant TFs must have
  (i) the same pole-zero deficit
Second Causality Condition

\[ C(z) = \frac{D_{ZAS}(z)}{N_{ol}(z)z^{-l}} \left[ \frac{N_{cl1}(z)z^{-l}}{D_{cl}(z) - N_{cl1}(z)z^{-l}} \right] \]

- Causal controller must have no time advance.
- Closed-loop and plant TFs must have (ii) the same time delay.
Non-minimum Phase Plant

• For a non-minimum phase plant zero at $\bar{z}$

$$G_{ZAS}(z) = G_1(z)(z - \bar{z}), \quad |\bar{z}| > 1$$

$$G_{cl}(z) = \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)}$$

$$= \frac{C(z)G_1(z)(z - \bar{z})}{1 + C(z)G_1(z)(z - \bar{z})}$$

• No unstable pole-zero cancellation for asymptotic stability.

• Closed-loop zero at $\bar{z}$
Unstable Plant

- For an unstable plant pole at $\bar{p}$
  $$G_{ZAS}(z) = \frac{G_1(z)}{z - \bar{p}}, \quad |\bar{p}| > 1$$

$$1 - G_{cl}(z) = 1 - \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)}$$

$$= \frac{1}{1 + C(z)\frac{G_1(z)}{z - \bar{p}}} = \frac{z - \bar{p}}{z - \bar{p} + C(z)G_1(z)}$$

- No unstable pole-zero cancellation for asymptotic stability.
- Zero at $\bar{p}$ for $1 - G_{cl}(z)$
Zero Steady-state Error

• For a unit step input, the steady-state output is

\[ y(\infty) = \lim_{z \to 1} (z - 1)G_{cl}(z) \frac{z}{z - 1} = G_{cl}(1) \]

• For zero steady-state error due to step

\[ G_{cl}(1) = 1 \]
Summary: $G_{cl}(Z)$ Constraints

- Closed-loop and plant: same pole-zero deficits and time delays.
- Unstable plant zeros must be closed-loop zeros ($G_{cl}(Z)$ zeros).
- Unstable plant poles must be zeros of $1 - G_{cl}(Z)$
- For zero steady-state error due to step, $G_{cl}(1) = 1$
Pole-zero Matching

• Analog filter with transfer function

\[ G_a(s) = K \frac{\prod_{i=1}^{m}(s - a_i)}{\prod_{j=1}^{n}(s - b_j)} \]

• Digital filter transfer function

\[ G(z) = \alpha K \frac{(z + 1)^{n-m-1} \prod_{i=1}^{m}(z - e^{a_i T})}{\prod_{j=1}^{n}(z - e^{b_j T})} \]

\( \alpha = \) constant selected for equal filter gains at a critical frequency (match DC gains for LPF)
MATLAB: Pole-Zero Matching

>> wn=5; zeta=0.5;
>> ga=tf([wn^2],[1,2*zeta*wn,wn^2]); % Analog TF
>> g=c2d(ga,.1,'matched') % Pole-zero matching

Transfer function:
0.09634 z + 0.09634
-------------------
z^2 - 1.414 z + 0.6065

Sampling time: 0.1
Procedure 6.3

• Select the desired settling time $T_s$ and the desired overshoot.

• Select a suitable continuous-time 1$^{\text{st}}$ or 2$^{\text{nd}}$ order closed-loop transfer function with unit gain.

• Obtain $G_{cl}(z)$ using pole-zero matching.

• Verify that $G_{cl}(z)$ meets the conditions for causality, stability, and steady-state error. If not, modify $G_{cl}(z)$ until the conditions are met.
Example

Design a digital controller for the DC motor speed control system with analog plant transfer function to obtain:

\[ G(s) = \frac{1}{(s + 1)(s + 10)} \]

(i) zero steady-state error due to a unit step, and (ii) a settling time of about 4 s.

Use \( T = 0.02 \) s.
Solution

\[ G_{ZAS}(z) = (1 - z^{-1}) \mathcal{Z}\left\{ \frac{G(s)}{s} \right\} \]

\[ = 1.8604 \times 10^{-4} \frac{z + 0.9293}{(z - 0.8187)(z - 0.9802)} \]

- No poles or zeros outside the unit circle.
- Pole-zero difference = 1.
- Choose closed-loop poles (2nd order) for \((\zeta = 0.88, \omega_n = 1.15 \text{ rad/s})\)
- Gain for zero steady-state error due to step.

\[ G(s) = \frac{1.322}{s^2 + 2.024s + 1.322} \]
MATLAB

• Desired closed-loop transfer function
  • Use pole-zero matching to get $G_{cl}(z)$
  • Zero at $-1$ for a pole-zero difference of 1 like $G_{ZAS}(z)$

```matlab
Ps=tf(1.322,[1 2.024 1.322])
Pz=c2d(Ps,0.02,'matched')
```
Solution (Cont.)

\[ G_{cl}(z) = 0.25921 \times 10^{-3} \frac{(z + 1)}{z^2 - 1.96z + 0.9603} \]

- Desired Controller (calculate with MATLAB)

\[ C(z) = \frac{1}{G_{ZAS}(z)} \left[ \frac{G_{cl}(z)}{1 - G_{cl}(z)} \right] \]

\[ C(z) = 1.3932 \frac{(z - 0.8187)(z - 0.9802)(z + 1)}{(z - 1)(z - 0.9601)(z + 0.2939)} \]
Step Response

System: gcl
Settling Time (sec): 3.71
Finite Settling Time

- **CT systems**: asymptotically (infinite time) settle at the desired output.
- **DT systems**: can settle at the reference output after a finite interval then follow it exactly.
- Finite settling time designs may exhibit undesirable inter-sample behavior and must be carefully checked before implementation.
- Use synthesis approach to obtain the desired controller for finite settling time.
**Deadbeat Controller**

- Finite settling time control
- Desired closed-loop transfer function
  \[ G_{cl}(z) = z^{-m} \]

\[
C(z) = \frac{1}{G_{ZAS}(z)} \left[ \frac{G_{cl}(z)}{1 - G_{cl}(z)} \right] = \frac{1}{G_{ZAS}(z)} \left[ \frac{z^{-m}}{1 - z^{-m}} \right]
\]

- Simple design: only design parameter is the sampling period \( T \)
Example 6.22

Design a deadbeat controller for the DC motor speed control system with transfer function

\[ G(s) = \frac{1}{(s + 1)(s + 10)} \]

1. Use a sampling period \( T=0.02 \text{ s} \).
2. Redesign the controller with \( T=0.1 \text{ s} \).
Solution

• Discretized process transfer function

\[
G_{ZAS}(z) = (1 - z^{-1})Z \left\{ \frac{G(s)}{s} \right\} \\
= 1.8604 \times 10^{-4} \frac{z + 0.9293}{(z - 0.8187)(z - 0.9802)}
\]

• No poles or zeros outside or on the unit circle.

• Can design a deadbeat controller.

\[
G_{cl}(z) = z^{-1}, \quad C(z) = \frac{1}{G_{ZAS}(z)} \left[ \frac{1}{z - 1} \right]
\]

\[
C(z) = 5375.0533 \frac{(z - 0.8187)(z - 0.9802)}{(z - 1)(z + 0.9293)}
\]
Sampled and analog step response for deadbeat control of Example 6.22 ($T=0.02$ s)
Control variable for deadbeat control of Example 6.22 \((T=0.02 \text{ s})\)
Deadbeat Control

• Large gain may saturate DAC.
• Controller causes inter-sample oscillations.
• Controller is much worse than the error at the sampling points would indicate.
Intersample Behavior

• Consider the system with analog output

Block diagram for finite settling time design with analog output.
Analog Output

- Use transfer function of ZOH.
- Use block diagram manipulation.

\[ E^*(s) = \frac{R^*(s)}{1 + C^*(s)(1 - e^{-sT}) \left( \frac{G(s)}{s} \right)^*} \]

\[ Y(s) = \left( \frac{1 - e^{-sT}}{s} \right) \frac{G(s)C^*(s)R^*(s)}{1 + C^*(s)G^*_ZAS(s)} \]
Step Response: $R(z) = 1/(1 - z^{-1})$

$$C(z) = 5375.0533 \frac{(z - 0.8187)(z - 0.9802)}{(z - 1)(z + 0.9293)} \frac{z + 0.9293}{(z - 0.8187)(z - 0.9802)}$$

$$G_{ZAS}(z) = 1.8604 \times 10^{-4} \frac{z + 0.9293}{(z - 0.8187)(z - 0.9802)}$$

$$Y(s) = \left( \frac{1 - e^{-sT}}{s} \right) \frac{G(s)C^*(s)R^*(s)}{1 + C^*(s)G_{ZAS}^*(s)}$$

$$Y(s) = \frac{5375.0533}{s(s + 1)(s + 10)} \frac{(1 - 0.8187e^{-sT})(1 - 0.9802e^{-sT})}{(1 + 0.9293e^{-sT})}$$
Analog Output Expansion

\[ Y(s) = \frac{5375.0533}{s(s + 1)(s + 10)} \left( 1 - 2.782e^{-sT} + 3.3378e^{-2sT} \right) \]

\[ -2.2993e^{-3sT} + 0.6930e^{-4sT} + \ldots \]

- Partial fraction expansion

\[ y(t) = 5375.0533 \]

\[ \left\{ \begin{array}{l}
\left( \frac{1}{10} - \frac{1}{9} e^{-t} + \frac{1}{90} e^{-10t} \right) 1(t) \\
-0.2782 \left( \frac{1}{10} - \frac{1}{9} e^{-(t-T)} + \frac{1}{90} e^{-10(t-T)} \right) 1(t-T) \\
+3.3378 \left( \frac{1}{10} - \frac{1}{9} e^{-(t-2T)} + \frac{1}{90} e^{-10(t-2T)} \right) 1(t-2T) \\
-2.2993 \left( \frac{1}{10} - \frac{1}{9} e^{-(t-3T)} + \frac{1}{90} e^{-10(t-3T)} \right) 1(t-3T) + \ldots
\end{array} \right. \]
Inter-sample oscillations with deadbeat control

- At the sampling points $y(kT) = 1$ but between samples the output oscillates wildly
Limitations of deadbeat control

1- Minimum phase transfer function $G_{ZAS}(z)$ (i.e. all zeros inside the unit circle), since its zeros are controller poles.

2- Controller may require excessively high gains that cause DAC saturation.

3- Intersample oscillations of system analog output.

Lesson from finite settling time designs:
Check analog output of digital control system for satisfactory intersample behavior.
Eliminating Intersample Oscillations

• Obtain the expression for the control.

\[ U(z) = \frac{Y(z)}{G_{ZAS}(z)} = \frac{Y(z)}{R(z)} \cdot \frac{R(z)}{G_{ZAS}(z)} = G_{cl}(z) \frac{R(z)}{G_{ZAS}(z)} \]

• Choose \( G_{cl}(z) \) and satisfy the gain condition for zero steady-state error (for step, \( G_{cl}(1) = 1 \))

\[ u(kT) = \text{constant}, \quad k \geq n \]

• Fix the control input after \( n \) sampling points for an \( n^{th} \) order system with zero error.

\[ e(kT) = 0, \quad k \geq n \]
Open-loop gain

\[ L(z) = C(z)G_{ZAS}(z) = \frac{N_L(z)}{D_L(z)} \]

\[ G_{cl}(z) = \frac{N_L(z)}{N_L(z) + D_L(z)} \]

- Fixed control and \( e(kT) = 0, k \geq n \) (type 1 system) if and only if
  i. \( N_L(z) + D_L(z) = z^n \)
  ii. \( D_L(z) \) has a pole at \( z = 1 \) (zero \( e(\infty) \) )
  iii. No pole-zero cancellation
Loop Gain Condition

• Loop gain

\[ L(z) = \frac{N_L(z)}{D_L(z)} \]

• Closed-loop transfer function

\[ G_{cl}(z) = \frac{L(z)}{1 + L(z)} = \frac{N_L(z)}{N_L(z) + D_L(z)} = \frac{N_L(z)}{z^n} \]

• Assume no pole-zero cancellation

\[ N_L(z) + D_L(z) = z^n \]
Minimum-phase Stable Plant

\[ G_{ZAS}(z) = \frac{N_{ZAS}(z)}{D(z)} = \frac{a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}{D(z)} \]

Case 1: Type 0 Plant, no pole at \( z = 1 \)

For zero steady-state error

\[ C(z) = \frac{N_C(z)}{D_C(z)} = K \frac{D(z)}{(z - 1)D_1(z)} \]

Case 2: Type I Plant, has a pole at \( z = 1 \)

No controller pole at \( z = 1 \)

\[ D(z) = (z - 1)D_1(z), \quad C(z) = K \frac{D_1(z)}{D_C(z)} \]
Loop Gain and $G_{cl}(z)$

- Loop Gain: $L(z) = \frac{N_L(z)}{D_L(z)} = C(z)G_{ZAS}(z)$

Case 1: $L(z) = K \frac{D(z)}{(z-1)D_1(z)} \times \frac{N_{ZAS}(z)}{D(z)}$

Case 2: $L(z) = K \frac{D_1(z)}{D_C(z)} \times \frac{N_{ZAS}(z)}{(z-1)D_1(z)}$

Both cases:

$$G_{cl}(z) = \frac{N_L(z)}{N_L(z) + D_L(z)} = \frac{KN_{ZAS}(z)}{z^n}$$
Zero Steady-state Error

\[ G_{cl}(z) = \frac{N_L(z)}{z^n} = K \frac{N_{ZAS}(z)}{z^n} \]

\[ N_{ZAS}(z) = a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \]

- For zero steady-state error

\[ G_{cl}(1) = K N_{ZAS}(1) = K \sum_{i=1}^{n-1} a_i = 1 \]

\[ K = \frac{1}{\sum_{i=1}^{n-1} a_i} \]
Controller: Case 1

\[ G_{ZAS}(z) = \frac{a_{n-1}z^{n-1} + \ldots + a_1z + a_0}{D(z)} \]

\[ C(z) = \frac{KD(z)}{(z-1)(z^{n-1} + \ldots + b_1z + b_0)} \]

- Closed-loop characteristic polynomial

\[ N_L(z) + D_L(z) = z^n \]

\[ = K[a_{n-1}z^{n-1} + \ldots + a_1z + a_0] \]

\[ +(z-1)(z^{n-1} + b_{n-2}z^{n-2} + \ldots + b_1z + b_0) \]

- Equate coefficients: solve for the controller.
Case 1: Design Equations

• Equating coefficients gives

\[ b_0 = Ka_0 \]
\[ b_i = b_{i-1} + Ka_i, \quad i = 2, \ldots, n - 2 \]
\[ C(z) = \frac{KD(z)}{(z - 1)(z^{n-1} + \cdots + b_1z + b_0)} \]
\[ K = \frac{1}{\sum_{i=1}^{n-1} a_i} \]
Controller: Case 2 (like Case 1)

- $G_{ZAS}(z)$ with a pole at $z = 1$

$$G_{ZAS}(z) = \frac{a_{n-1}z^{n-1} + \cdots + a_1z + a_0}{(z - 1)D(z)}$$

- $C(z) = K \frac{D(z)}{z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0}$

- $N_L(z) + D_L(z) = K \left[ a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \right] + \left( z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0 \right) = z^n$

- Equate coefficients to solve for the controller.
Case 2: Design Equations

- Equating coefficients gives
  \[ b_i = K a_i, \quad i = 0, \ldots, n - 1 \]
  \[ C(z) = \frac{KD(z)}{z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0} \]
  \[ K = \frac{1}{\sum_{i=1}^{n-1} a_i} \]
Stability Condition: Unstable Zeros

• Unstable plant zeros must be closed-loop zeros

\[
G_{cl}(z) = \frac{K N_{ZAS}(z)}{N_{ZAS}(z) + (z - 1)D_C(z)}
\]

• Unstable plant zeros must not cancel

• \( D_C(z) \) must not include the unstable plant zeros

\[
D_C(\bar{z}) \neq 0, \text{ if } N_{ZAS}(\bar{z}) = 0, |\bar{z}| > 1
\]

\[
\bar{z}^{n-1} + b_{n-2} \bar{z}^{n-2} + \cdots + b_1 \bar{z} + b_0 \neq 0
\]
Stability Condition: Unstable Poles

- Unstable plant poles must be zeros of

\[ 1 - G_{cl}(z) = \frac{(1 - K)N_{ZAS}(z) + (z - 1)D_C(z)}{N_{ZAS}(z) + (z - 1)D_C(z)} \]

- Unstable plant poles \( D_{ZAS}(\bar{p}) = 0, |\bar{p}| > 1 \), must not cancel:

\[ N_{ZAS}(\bar{p}) + (\bar{p} - 1)D_C(\bar{p}) \neq 0 \]

\[ (1 - K)N_{ZAS}(\bar{p}) + (\bar{p} - 1)D_C(\bar{p}) = 0 \]
Example 6.23

Design a ripple-free deadbeat controller for the positioning system with transfer function

\[ G(s) = \frac{1}{s(s + 1)} \]

The sampling period is chosen as \( T = 0.1 \)
Solution

• Discretized process transfer function

\[ G_{ZAS}(z) = (1 - z^{-1})Z \left\{ \frac{G(s)}{s} \right\} \]

\[ = 0.0048374 \frac{z + 0.9672}{(z - 1)(z - 0.9048)} \]

• 2\textsuperscript{nd} order Process: zero control after \( n = 2 \) samples

\[ N_{ZAS}(z) = a_0 + a_1 z = 0.004679 + 0.0048374z \]
Controller Design

\[ K = \frac{1}{a_0 + a_1} = 105.0833 \]

\[ C(z) = \frac{N_C(z)}{(z - 1)D_1(z)} = \frac{KD_{ZAS}(z)}{(z - 1)(b_0 + b_1z)} \]

\[ b_0 = Ka_0 = 105.0833 \times 4.679 \times 10^{-3} = 0.4917 \]

\[ b_1 = b_0 + Ka_1 = K(a_0 + a_1) = 1 \]

\[ C(z) = 105.0833 \frac{z - 0.9048}{z + 0.4917} \]

- Closed-loop transfer function

\[ G_{cl}(z) = \frac{KN_{ZAS}(z)}{z^2} = 0.50833 \left( \frac{z + 0.9672}{z^2} \right) \]
Step response for deadbeat control
Control variable for deadbeat control.
Example 6.25

Design a deadbeat controller for the isothermal chemical reactor

\[ G(s) = -\frac{1.1354(s - 2.818)}{(s + 2.672)(s + 2.47)} \]

With a sampling period \( T = 0.01 \text{s} \)
Solution

\[ G_{ZAS} = -\frac{0.01093(z - 1.029)}{(z - 0.9797)(z - 0.9736)} \]

- Open-loop stable, non-minimum phase

\[ N(z) = a_0 + a_1z = 0.01122 - 0.01091z \]

\[ K = \frac{1}{a_0 + a_1} = 3.2067 \times 10^3 \]

\[ b_0 = Ka_0 = 13.2067 \times 10^3 \times 0.01122 = 35.9799 \]

\[ b_1 = b_0 + Ka_1 = K(a_0 + a_1) = 1 \]
Equate Coefficients

\[ C(z) = \frac{KD(z)}{(z - 1)(b_0 + b_1z)} \]

\[ = 3200.34 \frac{(z - 0.9797)(z - 0.9736)}{(z - 1)(z + 35.9799)} \]

- Unstable controller and plant but closed-loop stable.

- Open-loop stable: No 1 \(-\) \(G_{cl}\) constraint

- Non-minimum phase: \(G_{cl} = -34.98 \frac{z-1.029}{z^2}\) includes the zero outside the unit circle.
Check Conditions

\[ G_{cl} = -34.98 \frac{z - 1.029}{z^2} \]

\[ 1 - G_{cl} = \frac{(z - 1)(z + 35.98)}{z^2} \]
Example 6.25: Step Response

Sampled and analog step response for deadbeat control
Example 6.25: Control Variable

Control variable for the deadbeat control