Time Response of a Discrete-Time System

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Outline

• Response of linear system.
• Convolution theorem.
• Impulse response.
• z-transfer function.
Convolution Summation

Impulse response sequence

The response of a discrete time system to a unit impulse.

- Obtain response for an arbitrary input sequence in terms of the impulse response.
  \[
  \{u(k)\} = \{u(0), u(1), \ldots, u(k), \ldots \}
  \]
Principle of Superposition

- Write input as a weighted sum of deltas

\[ u(k) = u(0)\delta(k) + u(1)\delta(k - 1) + \cdots + u(i)\delta(k - i) + \cdots \]

\[ = \sum_{i=0}^{\infty} u(i)\delta(k - i) \]

- Time Response: sum of impulse scaled impulse responses

\[ y(k) = u(0)h(k) + u(1)h(k - 1) + \cdots + u(i)h(k - i) + \cdots \]

\[ y(k) = \sum_{i=0}^{\infty} u(i)h(k - i) \]
Causal System

• Response to $\delta(k - i)$ starts at time $i$
  
  $$h(k - i) = 0, \quad i > k$$

  $$y(k) = u(0)h(k) + u(1)h(k - 1) + \cdots + u(k)h(0) + \cdots$$

  $$= \sum_{i=0}^{k} u(i)h(k - i)$$

• Change summation variable: $j = k - i$

  $$y(k) = u(k)h(0) + u(k - 1)h(1) + \cdots + u(0)h(k) + \cdots$$

  $$= \sum_{j=0}^{k} u(k - j)h(j)$$
Response of a LTI System

Theorem 2.2: The response of a LTI discrete time system \( \{y(k)\} \) to an arbitrary input sequence \( \{u(k)\} \) is given by the convolution summation of the input sequence and the impulse response sequence of the system \( \{h(k)\} \).

\[
y(k) = \sum_{i=0}^{\infty} u(i) h(k - i)
\]
Response of Causal LTI DT System to an Impulse at $iT$

\[ \delta(k - i) \]

\[ \{h(k-i)\} \]

......

LTI System
<table>
<thead>
<tr>
<th>Input</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(0)\delta(k)$</td>
<td>$u(0){h(k)}$</td>
</tr>
<tr>
<td>$u(1)\delta(k - 1)$</td>
<td>$u(1){h(k - 1)}$</td>
</tr>
<tr>
<td>$u(2)\delta(k - 2)$</td>
<td>$u(2){h(k - 2)}$</td>
</tr>
</tbody>
</table>
Example: $k = 2$

\[ y(2) = u(0)h(2) + u(1)h(1) + u(2)h(0) \]

\[ = \sum_{i=0}^{2} u(i)h(2 - i) \]
Z-Transfer Function

- \( H(z) = \) the z-transfer function or transfer function.
- The transfer function & the impulse response sequence are z-transform pairs.

\[ h(k) \overset{Z}{\leftrightarrow} H(z) \]
Convolution Theorem

- The z-transform of the convolution of two time sequences is equal to the product of their z –transforms (Proved earlier)
- Linear system: \( y(k) = h(k) * u(k) \)
- The z-transform of the output of a linear system is the product of the transfer function and the z –transform of the input. 
  \[ Y(z) = H(z)U(z) \]
Example 2.20

$$y(k + 1) - ay(k) = u(k), y(0) = 0$$

Find the impulse response $h(k)$:

(a) From the difference equation.

(b) Using z-transformation.
(a) From Difference Equation

• Let $u(k) = \delta(k)$
  \[ y(k + 1) - ay(k) = \delta(k), \quad y(0) = 0 \]
  \[ y(1) = 1 \]
  \[ y(2) = a y(1) = a \]
  \[ y(3) = ay(2) = a^2 \]

• By induction
  \[ h(i) = \begin{cases} 
  a^{i-1}, & i = 1, 2, 3, \ldots \\
  0, & i < 1 
  \end{cases} \]
(b) Using z-transformation

\[ y(k + 1) - ay(k) = u(k), \quad y(0) = 0 \]

\[
H(z) = \frac{Y(z)}{U(z)} = \frac{1}{z - a}
\]

\[
= z^{-1} \frac{z}{z - a}
\]

- Use the delay theorem

\[
h(i) = \begin{cases} 
a^{i-1}, & i = 1,2,3, \ldots \\
0, & i < 1 \end{cases}
\]
Output fo DT System

• Use the z-transform to find the system output:
  i. z-transform the input.
  ii. Multiply the z-transform of the input by the z-transfer function.
  iii. Inverse z-transform to obtain the output sequence.
Example 2.21

\[ y(k + 1) - y(k) = u(k + 1), \quad y(0) = 0 \]

Find the system transfer function and its response to a sampled unit step.

Solution: z-transform \( H(z) = \frac{z}{z-1} \)

\[ Y(z) = \left( \frac{z}{z-1} \right) \times \left( \frac{z}{z-1} \right) = z \frac{z}{(z-1)^2} \]

\[ y(i) = \begin{cases} 
   i + 1, & i = 0, 1, 2, 3, \ldots \\
   0, & i < 0
\end{cases} \]