# The $\mathcal{Z}$-transform

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# $\mathcal{Z}$-transform Definition

- **Definition 2.1** Given the causal sequence $\{u_0, u_1, u_2, \ldots, u_k, \ldots\}$ then its z-transform is defined as

$$U(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_k z^{-k} \ldots$$

$$= \sum_{k=0}^{\infty} u_k z^{-k}$$

$z^{-1}$=time delay operator

# Example

Obtain the z-transform of the sequence

$$\{u(k)\} = \{5,0,2,0,1,4,0,3,0,0,\ldots\}$$

*Solution* Definition 2.1 gives

$$U(z) = 5 + 2z^{-2} + z^{-4} + 4z^{-5} + 3z^{-7}$$

# $\mathcal{Z}$-transform Definition

**Definition 2.2** Laplace transform the impulse train representation of sampled signal

$$u^*(t) = u_0\delta(t) + u_1\delta(t-T) + \cdots + u_k\delta(t-kT) + \cdots$$

$$= \sum_{k=0}^{\infty} u_k\,\delta(t-kT)$$

$$U^*(s) = u_0 + u_1 e^{-sT} + \cdots + u_k e^{-skT} + \cdots$$

$$= \sum_{k=0}^{\infty} u_k\, e^{-ksT} = \sum_{k=0}^{\infty} u_k \left(e^{sT}\right)^{-k}$$

$$U(z) = \sum_{k=0}^{\infty} u_k\, z^{-k}, \qquad z = e^{sT}$$

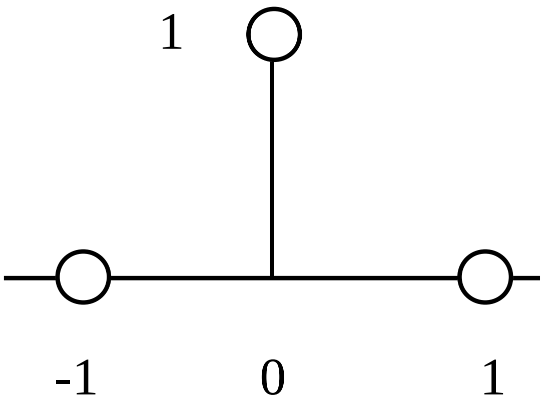
# Identities Used Repeatedly

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}, \qquad a \neq 1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1 - a}, \qquad |a| < 1$$

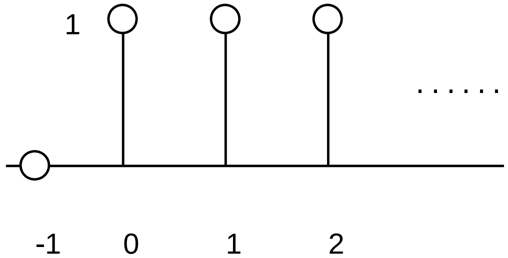
# Unit Impulse

$$u(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$




- Definition 2.1: $U(z) = 1$
- Impulse-sampled version:
  $u^*(t) = \delta(t)$, Laplace transform $U^*(s) = 1$
- z-transform obtained using Definition 2.2 same as Definition 2.1

# Sampled Unit Step



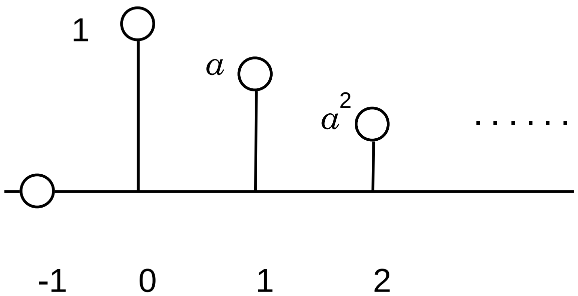

$$\{u(k)\}_{k=0}^{\infty} = \{1,1,1,1,\ldots$$

z-transform Definition 2.1

$$U(z) = 1 + z^{-1} + z^{-2} + \cdots + z^{-k} + \cdots = \sum_{k=0}^{\infty} z^{-k}$$

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

# Sampled Exponential




$$u(k) = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

z-transform Definition 2.1

$$U(z) = 1 + az^{-1} + a^2 z^{-2} + \cdots + a^k z^{-k} + \cdots$$

$$= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$U(z) = \frac{1}{1 - (a/z)} = \frac{z}{z - a}$$

# $\mathcal{Z}$-transform Properties

Linearity: Use Definition 2.2 and the linearity of the Laplace transform.

$$\mathcal{Z}\{\alpha f_1(k) + \beta f_2(k)\} = \alpha F_1(z) + \beta F_2(z)$$

Example

$$f(k) = 3 \times 1(k) + 2(0.1)^k, \qquad k = 0,1,2, \ldots$$

$$F(z) = \mathcal{Z}\{3 \times 1(k) + 2(0.1)^k\}$$

$$= 3\mathcal{Z}\{1(k)\} + 2\mathcal{Z}\{(0.1)^k\}$$

$$= \frac{3z}{z-1} + \frac{2z}{z-0.1} = \frac{z(5z-2.3)}{(z-1)(z-0.1)}$$

# Time Delay

- Use the time delay property of the Laplace transform $\mathscr{L}\{f(t-T_d)\} = e^{-sT_d}F(s)$

$$\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$$

Example $f(k) = \begin{cases} 0.2^{k-3}, k = 3, 4, 5, \ldots \\ 0, \quad elsewhere \end{cases}$

$$F(z) = \mathcal{Z}\{0.2^{k-3}\} = z^{-3}\mathcal{Z}\{0.2^k\}$$

$$= z^{-3} \times \frac{z}{z-0.2} = \frac{1}{z^2(z-0.2)}$$

# Time Advance

$$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$$

- Using proof by induction, generalize

$$\mathcal{Z}\{f(k+n)\}$$

$$= z^n F(z) - z^n f(0) - z^{n-1} f(1) - \cdots - zf(n-1)$$

# Time Advance: Proof

- Apply Defn. 2.1 to $f(k+1) = \{f(1), f(2), \ldots\}$

$$\mathcal{Z}\{f(k+1)\} = \sum_{k=0}^{\infty} f(k+1) z^{-k} = z \sum_{k=0}^{\infty} f(k+1) z^{-(k+1)}$$

$$= z\left\{\left[f(0) + \sum_{k=0}^{\infty} f(k+1) z^{-(k+1)}\right] - f(0)\right\}$$

- Change index of summation to $m = k + 1$

$$\mathcal{Z}\{f(k+1)\} = \left[z \sum_{m=0}^{\infty} f(m) z^{-m}\right] - zf(0) = zF(z) - zf(0)$$

# Example

Use the time advance property to find the z-transform of the causal sequence

$$\{f(k)\} = \{4,8,16,\dots\}$$

Solution $\quad f(k) = 2^{k+2} = g(k+2),$

$$g(k) = 2^k, \qquad k = 0,1,2,\dots$$

$$F(z) = z^2 G(z) - z^2 g(0) - z g(1)$$

$$= z^2 \frac{z}{z-2} - z^2 - 2z = \frac{4z}{z-2}$$

**Easier solution:**

- Write the sequence as $\{f(k)\} = 4\{1,2,4,\dots\} = 4\{2^k\}$
- Use the linearity of the *z*-transform.

# Discrete-Time Convolution

$$\mathcal{Z}\{f_1(k) * f_2(k)\} = \mathcal{Z}\left\{\sum_{i=0}^{k} f_1(i) f_2(k-i)\right\} = F_1(z) F_2(z)$$

Proof: Let the convolution give $\{y(k)\}$

$$Y(z) = y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots + y(k)z^{-k} + \cdots$$

$$= \sum_{k=0}^{\infty} y(k) z^{-k} = \sum_{k=0}^{\infty}\left[\sum_{i=0}^{k} f_1(i) f_2(k-i)\right] z^{-k}$$

- Causal Sequence $f(k-i) = 0, k < i$

# DT Convolution Proof (cont.)

$$Y(z) = \sum_{i=0}^{\infty} f_1(i) \left[ \sum_{k=i}^{\infty} f_2(k-i) z^{-k} \right]$$

Causal Sequence $f(k-i) = 0, k < i$

Change summation index from $k$ to $j = k - i$

$$Y(z) = \sum_{i=0}^{\infty} f_1(i) \left[ \sum_{j=0}^{\infty} f_2(j) z^{-(j+i)} \right]$$

$$= \sum_{i=0}^{\infty} f_1(i) z^{-i} \left[ \sum_{j=0}^{\infty} f_2(j) z^{-j} \right] = F_1(z) F_2(z)$$

# Example

Find the z-transform of the convolution of two sampled step sequences.

Solution:

Sampled step

$$\{f(k)\} = \{1,1,1,\dots\}, \qquad F(z) = \frac{z}{z-1}$$

By the convolution theorem,

*z*-transform = product of the *z*-transforms of two step sequences.

$$F(z) = \left(\frac{z}{z-1}\right) \times \left(\frac{z}{z-1}\right) = \left(\frac{z}{z-1}\right)^2$$

# Multiplication by Exponential

$$\mathcal{Z}\left\{a^{-k} f(k)\right\} = F(az)$$

Proof

$$LHS = \sum_{k=0}^{\infty} a^{-k} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) (az)^{-k}$$

$$= F(az)$$

# Example

Find the z-transform of the exponential sequence $f(k) = e^{-\alpha kT}, \quad k = 0, 1, 2, \ldots$

$$f(k) = \left(e^{\alpha T}\right)^{-k} \times 1, \qquad k = 0, 1, 2, \ldots$$

- $z$-transform of a sampled step

$$F(z) = \left(1 - z^{-1}\right)^{-1}$$

$$\mathcal{Z}\left\{\left(e^{\alpha T}\right)^{-k} \mathbf{1}(k)\right\} = \left(1 - \left(e^{\alpha T} z\right)^{-1}\right)^{-1}$$

$$= \frac{z}{z - e^{-\alpha T}}$$

(same as earlier example $a^{-k}, a = e^{\alpha T}$)

# Complex Differentiation

$$\mathcal{Z}\{k^m f(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$$

Proof (Induction)

(i) Establish validity for $m = 1$.

(ii) Assume validity for any $m$ and prove it for $m + 1$.

For $m = 1$, we have

$$\mathcal{Z}\{kf(k)\} = \sum_{k=0}^{\infty} kf(k)z^{-k} = \sum_{k=0}^{\infty} f(k)\left(-z\frac{d}{dz}\right)z^{-k}$$

$$= \left(-z\frac{d}{dz}\right)\sum_{k=0}^{\infty} f(k)z^{-k} = \left(-z\frac{d}{dz}\right)F(z)$$

# Proof (Cont.)

- For any $m$, define $f_m(k) = k^m f(k), k = 0,1,2,\ldots$
- Assume $F_m(z) = \mathcal{Z}\{k^m f(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$

$$\mathcal{Z}\{k f_m(k)\} = \sum_{k=0}^{\infty} k f_m(k) z^{-k} = \sum_{k=0}^{\infty} f_m(k)\left(-z\frac{d}{dz}\right) z^{-k}$$

$$= \left(-z\frac{d}{dz}\right)\sum_{k=0}^{\infty} f_m(k) z^{-k} = \left(-z\frac{d}{dz}\right) F_m(z)$$

- Substitute for $F_m(z)$

$$\mathcal{Z}\{k^{m+1} f(k)\} = \mathcal{Z}\{k f_m(k)\} = \left(-z\frac{d}{dz}\right)^{m+1} F(z)$$

# Example

Find the z-transform of the sampled ramp sequence

$$f(k) = k,\ k = 0, 1, 2, \ldots$$

Solution: $z$-transform of a sampled step $F(z) = \frac{z}{z-1}$

Write $f(k)$ as: $f(k) = k \times 1,\ k = 0, 1, 2, \ldots$

Apply the complex differentiation property

$$\mathcal{Z}\{k \times 1\} = \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right)$$

$$= (-z)\frac{(z-1) - z}{(z-1)^2} = \frac{z}{(z-1)^2}$$

Note: For the transform of $kT$ multiply by $T$.

# Inversion of the z-Transform

1. Long division: gives as many terms of series as desired.
2. Partial fraction expansion and table look-up: similar to Laplace transform inversion.

# Long Division

(i) Using long division, expand $F(z)$ as a series

$$F_t(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots + f_i z^{-i}$$

$$= \sum_{k=0}^{i} f_k z^{-k}$$

(*ii*) Write the inverse transform as the sequence

$$\{f_0, f_1, \dots, f_i, \dots\}$$

# Example

Inverse z-transform $F(z) = \frac{z+1}{z^2+0.2z+0.1}$

*Solution:*

(i) Long Division

$$
\begin{array}{r|l}
 & z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \cdots \\
\hline
z^2 + 0.2z + 0.1 & z + 1 \\
 & z + 0.2 + 0.1z^{-1} \\
\hline
 & \quad 0.8 - 0.10z^{-1} \\
 & \quad 0.8 - 0.16z^{-1} + 0.08z^{-2} \\
\hline
 & \qquad -0.26z^{-1}
\end{array}
$$

$$F_t(z) = 0 + z^{-1} + 0.8z^{-2} + (-0.26)z^{-3}$$

(ii) Inverse Transformation $\{f(k)\} = \{0,1,0.8,-0.26, \ldots\}$

# Partial Fraction Expansion

(i) Find the partial fraction expansion of $F(z)/z$.

(ii) Obtain the inverse transform $f(k)$ using the z-transform tables.

**Three types of z-domain functions $F(z)$:**

1. $F(z)$ with simple (non-repeated) real poles.
2. $F(z)$ with complex conjugate & real poles.
3. $F(z)$ with repeated poles.

# I: Simple Real Roots

- Residue of a complex function $F(z)$ at a simple pole $z_i$

$$A_i = (z - z_i)F(z)]_{z \to z_i}$$

- Residue = partial fraction coefficient of the $i^{th}$ term of the expansion

$$F(z) = \sum_{i=1}^{n} \frac{A_i}{z - z_i}$$

# Example

- Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2+0.3z+0.02}$$

Solution: Solve using two different methods.

(i) Partial Fraction Expansion (dividing by *z*)

$$\frac{F(z)}{z} = \frac{z+1}{z(z+0.1)(z+0.2)} = \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$$

$$F(z) = \frac{Az}{z} + \frac{Bz}{z+0.1} + \frac{Cz}{z+0.2}$$

# Example (cont.)

$$A = z\frac{F(z)}{z}\bigg|_{z=0} = F(0) = \frac{1}{0.02} = 50$$

$$B = (z+0.1)\frac{F(z)}{z}\bigg|_{z=-0.1} = \frac{1-0.1}{(-0.1)(0.1)} = -90$$

$$C = (z+0.2)\frac{F(z)}{z}\bigg|_{z=-0.2} = \frac{1-0.2}{(-0.2)(-0.1)} = 40$$

- Partial fraction expansion then multiply by $z$

$$F(z) = 50 - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

# MATLAB

$G(z)$ numerator $\mathrm{z} + 1$,

denominator $z^3 + 0.3z^2 + 0.02z$

**>> num = [1, 1]**

**>> den = [1, 0.3, 0.02, 0]**

- Partial Fraction Coefficients

**>> [r, p, k] = residue( num, den)**

- **p** = poles, **r** = residues
- **k** = coefficients of remainder polynomial.

# Results

```
r =
   40.0000
  -90.0000
   50.0000

p =
   -0.2000
   -0.1000
         0

k =
     []
```

# Example (cont.)

$$F(z) = 50 - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

- (*ii*) Table Lookup

$$f(k) = \begin{cases} 50\delta(k) - 90(-0.1)^k + 40(-0.2)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

- Note: $f(0) = 50 - 90 + 40 = 0$
- The time sequence can be rewritten as

$$f(k) = \begin{cases} -90(-0.1)^k + 40(-0.2)^k, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

# Example (cont.)

(i) Partial Fraction Expansion (without dividing by *z*)

$$F(z) = \frac{z+1}{(z+0.1)(z+0.2)} = \frac{A}{z+0.1} + \frac{B}{z+0.2}$$

- Partial fraction coefficients

$$A = (z+0.1)F(z)]_{z=-0.1} = \frac{1-0.1}{0.1} = 9$$

$$B = (z+0.2)F(z)]_{z=-0.2} = \frac{1-0.2}{-0.1} = -8$$

# Example (cont.)

- Partial Fraction Expansion

$$F(z) = \frac{9}{z+0.1} - \frac{8}{z+0.2}$$

$$= \frac{9z}{z+0.1} z^{-1} - \frac{8z}{z+0.2} z^{-1}$$

(ii) Table Lookup (use the delay theorem)

$$f(k) = \begin{cases} 9(-0.1)^{k-1} - 8(-0.2)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

(Verify: same answer as before)

# II: Complex Conjugate & Simple Real Roots

- Use the following z-transforms ($\omega_d$ rad.)

$$\mathcal{Z}\{e^{-\alpha k}\sin(k\omega_d)\} = \frac{e^{-\alpha}\sin(\omega_d)\,z}{z^2 - 2e^{-\alpha}\cos(\omega_d)\,z + e^{-2\alpha}}$$

$$\mathcal{Z}\{e^{-\alpha k}\cos(k\omega_d)\} = \frac{z[z - e^{-\alpha}\cos(\omega_d)]}{z^2 - 2e^{-\alpha}\cos(\omega_d)\,z + e^{-2\alpha}}$$

- Same denominator with complex conjugate roots:

$$z_{1,2} = e^{-\alpha}e^{\pm j\omega_d}$$

# Residues With Complex Conjugate Poles

$$F(z) = \frac{Az}{z-p} + \frac{A^*z}{z-p^*}$$

$$f(k) = A.p^k + A^*.p^{*k}$$

$$= |A|.|p|^k \left[e^{j(\theta_p k+\theta_A)} + e^{-j(\theta_p k+\theta_A)}\right]$$

$\theta_p(\theta_A)$= angle of pole $p$ ( partial fraction coefficient $A$)

- *Use:* $\cos(x) = \frac{e^{jx}+e^{-jx}}{2}$

$$f(k) = 2|A|.|p|^k \cos(\theta_p k + \theta_A)$$

# Combined Form

- Combine the complex conjugate terms

$$\frac{A}{z-p}+\frac{A^*}{z-p^*}=\frac{2Re\{A\}\,z-2Re\{A\,p^*\}}{z^2-(p+p^*)z+|p|^2}$$

Use

$$A+A^*=2Re\{A\}$$

$$Ap^*+A^*p=2Re\{A\,p^*\}$$

- Inverse transform: Use transform of sinusoid and equate coefficient.

# Example

Find the inverse z-transform of $F(z) = \frac{z^3+2z+1}{(z-0.1)(z^2+z+0.5)}$

Solution (*i*) *Partial Fraction Expansion*

- Dividing by *z* gives

$$\frac{F(z)}{z} = \frac{z^3 + 2z + 1}{z(z-0.1)(z^2+z+0.5)}$$

$$= \frac{A_1}{z} + \frac{A_2}{z-0.1} + \frac{Az+B}{z^2+z+0.5}$$

$$A_1 = F(0) = -20,$$

$$A_2 = (z-0.1)\frac{F(z)}{z}\bigg]_{z=0.1} \approx 19.689$$

# Example (cont.)

- Multiply by the denominator & equate coefficients

$$z^3: \quad A_1 + A_2 + A = 1$$

$$z^1: \quad 0.4\, A_1 + 0.5\, A_2 - 0.1\, B = 2$$

$A_1 = -20$, $A_2 =$19.689 (known), solve for $A$ and $B$

$$A \approx 1.311, \qquad B \approx -1.557$$

- **Check calculations**

$$z^0: -0.05A_1 = -0.05(-20) = 1$$

$$z^2: 0.9A_1 + A_2 - 0.1A + B$$

$$= 0.9(-20)+19.689-0.1(1.311) - 1.557 \approx 0$$

- Partial fraction expansion (multiply eqn. by $z$)

$$F(z) = -20 + \frac{19.689z}{z - 0.1} + \frac{1.311z^2 - 1.557z}{z^2 + z + 0.5}$$

# Example (cont.)

*(ii) Table Lookup* (1st two terms already known)

$$\frac{1.311z^2 - 1.557z}{z^2 - 2(-0.5)z + 0.5} = \frac{1.311z[z - e^{-\alpha}\cos(\omega_d)] - Ce^{-\alpha}\sin(\omega_d)\, z}{z^2 - 2e^{-\alpha}\cos(\omega_d)\, z + e^{-2\alpha}}$$

- Equate denominator coefficients

$$e^{-\alpha} = \sqrt{0.5} = 0.707$$

$$\cos(\omega_d) = -0.5/e^{-\alpha} = -\sqrt{0.5} = -0.707$$

$\omega_d = \frac{3\pi}{4} rad$, angle in 2nd quadrant, $\sin(\omega_d) = 0.707$

- Equate $z^1$ coefficients in the numerator

$$-1.311e^{-\alpha}\cos(\omega_d) - Ce^{-\alpha}\sin(\omega_d) = -0.5(C - 1.311) = -1.557$$

$$\Rightarrow C = 4.426$$

# Example (Cont.)

- Substitute for the coefficients

$$F(z) = -20 + \frac{19.689z}{z - 0.1}$$

$$+ \frac{1.311z[z - (0.707)\cos(3\pi/4)] - 4.426(0.707)\sin(3\pi/4)\,z}{z^2 - 2(0.707)\cos(3\pi/4)\,z + 0.5}$$

$$\mathcal{Z}\{e^{-\alpha k}\cos(k\omega_d)\} = \frac{z[z - e^{-\alpha}\cos(\omega_d)]}{z^2 - 2e^{-\alpha}\cos(\omega_d)\,z + e^{-2\alpha}}$$

$$\mathcal{Z}\{e^{-\alpha k}\sin(k\omega_d)\} = \frac{e^{-\alpha}\sin(\omega_d)\,z}{z^2 - 2e^{-\alpha}\cos(\omega_d)\,z + e^{-2\alpha}}$$

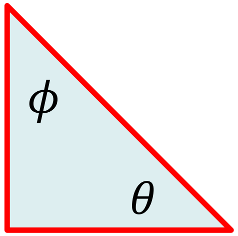
# Example (cont.)

- z-transform tables give $\left(e^{-\alpha k} = (e^{-\alpha})^k\right)$

$$f(k) = -20\delta(k) + 19.689(0.1)^k$$

$$+(0.707)^k[1.311\cos\left(\frac{3\pi k}{4}\right) - 4.426\sin\left(\frac{3\pi k}{4}\right), k \geq 0$$

$$4.616 = \sqrt{1.311^2 + 4.426^2}$$




$$\theta = \sin^{-1}\left(\frac{1.311}{4.426}\right) \approx 0.288$$

$$\phi = \cos^{-1}\left(\frac{1.311}{4.426}\right) \approx 1.283 = \frac{\pi}{2} - 0.288$$

# Use Trig. Identities

$$f(k) = -20\delta(k) + 19.689(0.1)^k$$

$$+4.616(0.707)^k \left[\frac{1.311}{4.616}\cos\left(\frac{3\pi k}{4}\right) - \frac{4.426}{4.616}\sin\left(\frac{3\pi k}{4}\right)\right], k \geq 0$$

$$\sin(A - B) = \sin(A)\cos(B) - \sin(B)\cos(A)$$

$$f(k) = -20\delta(k) + 19.689(0.1)^k$$

$$+4.616(0.707)^k \sin(3\pi k/4 - 0.288)$$

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$f(k) = -20\delta(k) + 19.689(0.1)^k$$

$$+4.616(0.707)^k \cos(3\pi k/4 + 1.283)$$

# Example (cont.)

*Residue Approach*

(i) *Partial Fraction Expansion*

Dividing by $z$ gives

$$\frac{F(z)}{z} = \frac{z^3 + 2z + 1}{z(z - 0.1)[(z + 0.5)^2 + 0.5^2]}$$

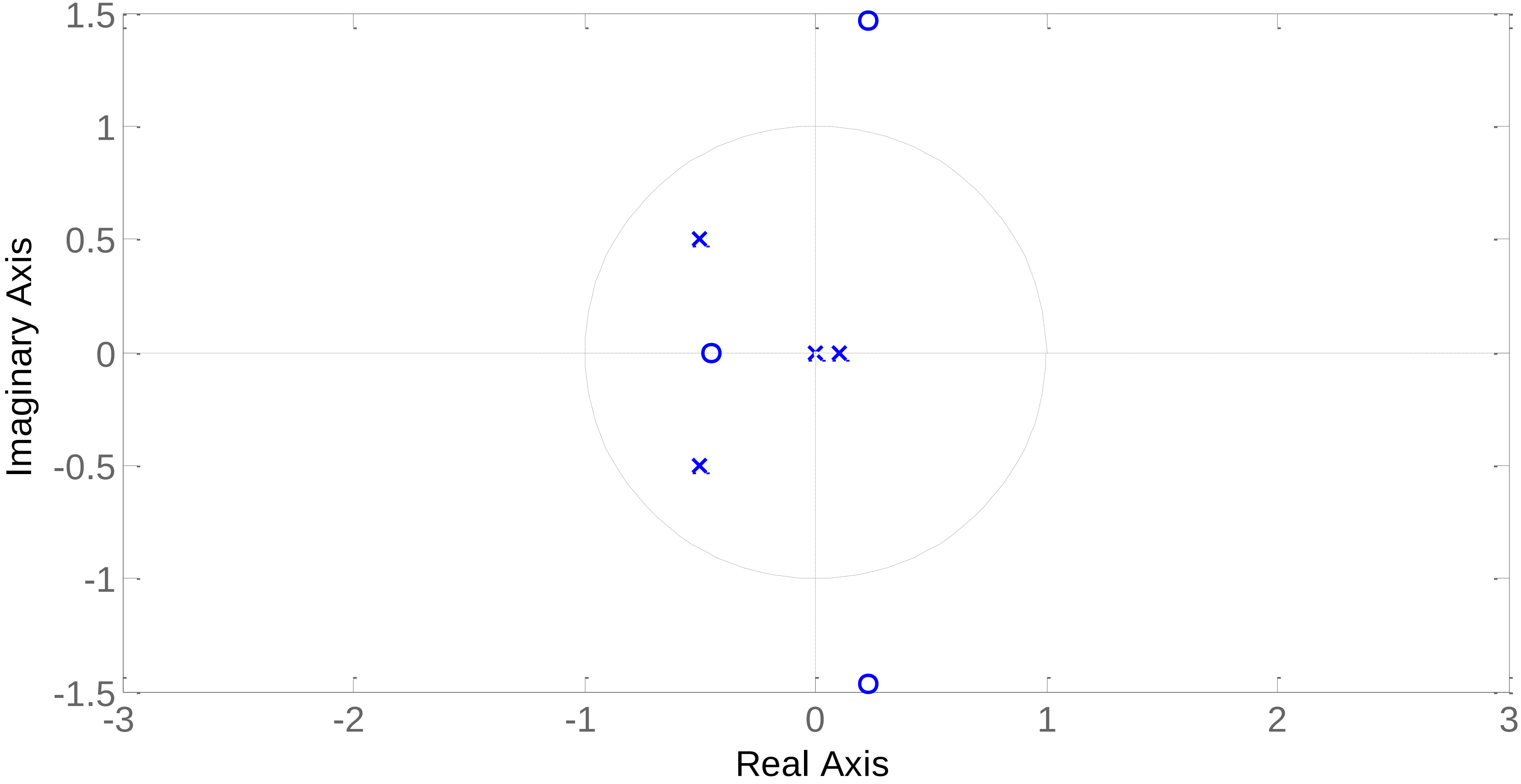
$$= \frac{A_1}{z} + \frac{A_2}{z - 0.1} + \frac{A_3}{z + 0.5 - j0.5} + \frac{A_3^*}{z + 0.5 + j0.5}$$

Obtain partial fraction expansion as in 1st approach

$$A_3 = \left.\frac{z^3 + 2z + 1}{z(z - 0.1)(z + 0.5 + j0.5)}\right]_{z=-0.5+j0.5}$$

$$\approx 0.656 + j2.213$$

# Pole-Zero Map



# Example (cont.)

$$F(z) = -20 + \frac{19.689z}{z-0.1} + \frac{(0.656+j2.213)z}{z+0.5-j0.5} + \frac{(0.656-j2.213)z}{z+0.5+j0.5}$$

- Convert $A_3$ from Cartesian to polar form

$$A_3 = 0.656 + j2.213 = 2.308e^{j1.283}$$

$$f(k) = 2|A||p|^k \cos\left(\theta_p k + \theta_A\right)$$

- Inverse z-transform to obtain

$$f(k) = -20\delta(k) + 19.689(0.1)^k + 4.616(0.707)^k \cos\left(\frac{3\pi k}{4} + 1.283\right)$$

as obtained earlier.

# MATLAB

$G(z)$ numerator $5(z+3)$, denominator $z^3 + 0.1z^2 + 0.4z$

**>> num = 5*[1, 3]**

**>> den = [1, 0.1, 0.4, 0]**

**>> denp =conv(den1, den 2)** % Multiply polynomials

- Partial Fraction Coefficients

**>> [r, p, k] = residue( num, den)**

- **p** = poles, **r** = residues , **k** = coefficients of the polynomial resulting from dividing the numerator by the denominator.

# MATLAB Example

```
>> [r,p,k]=residue(num,den)

r =

   0.6557 + 2.2131i
   0.6557 - 2.2131i
  19.6885
 -20.0000

p =

  -0.5000 + 0.5000i
  -0.5000 - 0.5000i
   0.1000
        0
```

$$\frac{F(z)}{z} = \frac{z^3+2z+1}{z(z-0.1)(z^2+z+0.5)}$$

# Other Form

```
>> A=2*real(r(1))
A =
    1.3115
>> B=2*real(r(1)*p(2))
B =
    1.5574
```

# III: Repeated Roots

$$F(z) = \frac{N(z)}{(z - z_1)^r \prod_{j=r+1}^{n} (z - z_j)}$$

$$= \sum_{i=1}^{r} \frac{A_{1i}}{(z - z_1)^{r+1-i}} + \sum_{j=r+1}^{n} \frac{A_j}{z - z_j}$$

$$A_{1i} = \frac{1}{(i-1)!} \frac{d^{i-1}}{dz^{i-1}} (z - z_1)^r F(z) \Big|_{z \to z_1}$$

$$i = 1, 2, \ldots, r$$

# Example

Obtain the inverse z-transform of the function

$$F(z) = \frac{1}{z^2(z-0.5)}$$

## Solution

(*i*) *Partial Fraction Expansion* (Dividing by *z*)

$$\frac{F(z)}{z} = \frac{1}{z^3(z-0.5)} = \frac{A_{11}}{z^3} + \frac{A_{12}}{z^2} + \frac{A_{13}}{z} + \frac{A_4}{z-0.5}$$

$$A_4 = (z-0.5)\frac{F(z)}{z}\bigg|_{z=0.5} = \frac{1}{z^3}\bigg|_{z=0.5} = 8$$

# Partial Fraction Coefficients

$$A_{11} = z^3 \frac{F(z)}{z}\bigg|_{z=0} = \frac{1}{z-0.5}\bigg|_{z=0} = -2$$

$$A_{12} = \frac{1}{1!}\frac{d}{dz} z^3 \frac{F(z)}{z}\bigg|_{z=0} = \frac{d}{dz}\frac{1}{z-0.5}\bigg|_{z=0}$$

$$= \frac{-1}{(z-0.5)^2}\bigg|_{z=0} = -4$$

$$A_{12} = \frac{1}{2!}\frac{d^2}{dz^2} z^3 \frac{F(z)}{z}\bigg|_{z=0} = \left(\frac{1}{2}\right)\frac{d}{dz}\frac{-1}{(z-0.5)^2}\bigg|_{z=0}$$

$$= \left(\frac{1}{2}\right)\frac{(-1)(-2)}{(z-0.5)^3}\bigg|_{z=0} = -8$$

# Example (cont.)

- Partial Fraction Expansion

$$F(z) = \frac{1}{z^2(z-0.5)} = \frac{8z}{z-0.5} - 2z^{-2} - 4z^{-1} - 8$$

- (*ii*) *Table Lookup*
- z-transform tables and Definition 2.1 yield

$$f(k) = \begin{cases} 8(0.5)^k - 2\delta(k-2) - 4\delta(k-1) - 8\delta(k), & k \geq 0 \\ 0, & k < 0 \end{cases}$$

# Example (cont.)

$$f(k) = 8(0.5)^k - 2\delta(k-2) - 4\delta(k-1) - 8\delta(k)$$

- Evaluating $f(k)$ at $k = 0, 1, 2,$ yields

$$f(0) = 8 - 8 = 0$$
$$f(1) = 8(0.5) - 4 = 0$$
$$f(2) = 8(0.5)^2 - 2 = 0$$
$$f(k) = \begin{cases} (0.5)^{k-3}, & k \geq 3 \\ 0, & k < 3 \end{cases}$$

- Using the delay theorem gives the same answer

$$F(z) = \frac{z}{z - 0.5} z^{-3}$$

# The Final Value Theorem

## Theorem 2.1 The Final Value Theorem

If a sequence approaches a constant limit as $k$ tends to infinity, then the limit is given by

$$f(\infty) = \lim_{k \to \infty} f(k)$$
$$= \lim_{z \to 1} (z - 1) F(z)$$

# Limitations of Final Value

Limit must exist for final value theorem to apply.

Does not apply to:

(i) An unbounded sequence.

(ii) An oscillatory sequence.

# Proof of Final Value Thm.

Let $f(k)$ have a constant limit as $k$ tends to infinity

$$f(k) = f(\infty) + g(k), \qquad k = 0,1,2,\ldots$$

$g(k)$= sequence that decays to zero as $k \to \infty$

$$\mathop{Lim}_{k\to\infty} f(k) = f(\infty)$$

$$F(z) = \frac{f(\infty)\, z}{z-1} + G(z)$$

- Partial fraction coefficient

$$f(\infty) = \mathop{Lim}_{z\to 1} (z-1)\frac{F(z)}{z}$$

# Example 2.17

Verify the final value theorem using the *z*-transform of a decaying exponential sequence and its limit as $k$ tends to infinity.

Solution: *z*-transform pair $\left\{e^{-akT}\right\} \overset{\mathcal{Z}}{\leftrightarrow} \frac{z}{z-e^{-aT}}$

- Limit with $a > 0$

$$f(\infty) = \underset{k\to\infty}{\mathcal{Lim}}\, e^{-akT} = 0$$

- Final value theorem

$$f(\infty) = \underset{z\to 1}{\mathcal{Lim}} \left(\frac{z-1}{z}\right)\left(\frac{z}{z-e^{-aT}}\right) = 0$$

# Example 2.18

Obtain the final value for the sequence whose *z*-transform is

$$F(z) = \frac{z^2(z-a)}{(z-1)(z-b)(z-c)}$$

What can you conclude concerning the constants $b$ and $c$ if it is known that the limit exist?

- Solution: Conditions for the validity of the final value theorem $|b| < 1,\ |c| < 1$
- Apply the final value theorem

$$f(\infty) = \lim_{z\to 1} \frac{(z-1)z^2(z-a)}{(z-1)(z-b)(z-c)} = \frac{1-a}{(1-b)(1-c)}$$

# Z-transform Solution of Difference Equations

Example 2.19: Solve the linear difference equation

$$x(k+2) - (3/2)x(k+1) + (1/2)x(k) = 1(k)$$

with the initial conditions $x(0) = 1, x(1) = 5/2$

Solution

(i) z-transform

$$\left[z^2X(z) - z^2x(0) - zx(1)\right] - \left(\frac{3}{2}\right)[zX(z) - zx(0)] + \left(\frac{1}{2}\right)X(z) = \frac{z}{z-1}$$

# (ii) Solve for X(z)

$$\left[z^2 - \left(\frac{3}{2}\right)z + \left(\frac{1}{2}\right)\right]X(z)$$

$$= \frac{z}{z-1} + z^2 + \left(\frac{5}{2} - \frac{3}{2}\right)z$$

$$X(z) = \frac{z[1 + (z+1)(z-1)]}{(z-1)(z-1)(z-0.5)}$$

$$= \frac{z^3}{(z-1)^2(z-0.5)}$$

# (*iii*) *Partial fraction expansion*

The partial fraction of $X(z)/z$ is

$$\frac{X(z)}{z}=\frac{z^2}{(z-1)^2(z-0.5)}=\frac{A_{11}}{(z-1)^2}+\frac{A_{12}}{z-1}+\frac{A_3}{z-0.5}$$

$$A_{11}=(z-1)^2\left.\frac{X(z)}{z}\right|_{z=1}=\left.\frac{z^2}{z-0.5}\right|_{z=1}$$

$$=\left.\frac{1}{1-0.5}\right|_{z=1}=2$$

$$A_3=(z-0.5)\left.\frac{X(z)}{z}\right|_{z=0.5}=\left.\frac{z^2}{(z-1)^2}\right|_{z=0.5}$$

$$=\frac{(0.5)^2}{(0.5-1)^2}=1$$

# Equating Coefficients

$$\frac{X(z)}{z} = \frac{z^2}{(z-1)^2(z-0.5)} = \frac{A_{11}}{(z-1)^2} + \frac{A_{12}}{z-1} + \frac{A_3}{z-0.5}$$

- Multiply by the denominator

$$z^2 = A_{11}(z-0.5) + A_{12}(z-0.5)(z-1) + A_3(z-1)^2$$

- Equate coefficient of $z^2$ $(A_3 = 1)$

$$z^2 : 1 = A_{12} + A_3 = A_{12} + 1, \qquad i.e.\ A_{12} = 0$$

$$X(z) = \frac{2z}{(z-1)^2} + \frac{z}{z-0.5}$$

*(iv) Inverse z-transform* (tables) $x(k) = 2k + (0.5)^k$

# Plot of the Solution $x(k)$