We recall the definition of a $k$-graph $\Lambda$ and of its $C^*$-algebra.

We define the cubical cohomology $H^n(\Lambda, G)$ with coefficients in an abelian group $G$.

If $\alpha$ is an automorphism of a $k$-graph $\Lambda$, we prove the existence of a long exact sequence relating the cohomology of $\Lambda$ and the cohomology of the $(k + 1)$-graph $\Lambda \times_\alpha \mathbb{Z}$.

The main result shows that if we twist the $C^*$-algebra of $\Lambda \times_\alpha \mathbb{Z}$ by a 2-cocycle $\varphi$, then $C^*_\varphi(\Lambda \times_\alpha \mathbb{Z})$ is isomorphic to the crossed product of a twisted $C^*$-algebra of $\Lambda$ by a modified automorphism.

We illustrate our results with an example.
A $k$-graph is a countable small category $\Lambda$ equipped with a functor $d : \Lambda \rightarrow \mathbb{N}^k$ satisfying the factorization property: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu \nu$.

For $n \in \mathbb{N}^k$, we write $\Lambda^n$ for $d^{-1}(n)$. Let $e_1, \ldots, e_k$ be the standard generators of $\mathbb{N}^k$.

$\Lambda$ is said to be row-finite with no sources if $0 < |v \Lambda^n| < \infty$ for all $v, n$.

An automorphism is a functor $\alpha : \Lambda \rightarrow \Lambda$ which preserves the degree and is invertible.

Note that an automorphism of $\Lambda$ determines automorphisms of $\Lambda^n$ for all $n \in \mathbb{N}^k$. 
For $r \geq 0$ the set of $r$-cubes is

$$Q_r(\Lambda) = \{ \lambda \in \Lambda : d(\lambda) \leq (1, 1, \ldots, 1), \sum_i d(\lambda)_i = r \}.$$ 

Then $Q_0(\Lambda) = \Lambda^0$ is the set of vertices, $Q_1(\Lambda) = \bigcup_{i=1}^{k} \Lambda^{e_i}$ is the set of edges, $Q_2(\Lambda)$ is the set of squares, etc.

When $1 \leq r \leq k$, each element $\lambda \in Q_r(\Lambda)$ has $2r$ faces $F^l_j(\lambda) \in Q_{r-1}(\Lambda)$.

The cubical homology of $\Lambda$ is identified with the homology of the complex $(\mathbb{Z}Q_\bullet, \partial_\bullet)$ where the boundary map $\partial_r : \mathbb{Z}Q_r \to \mathbb{Z}Q_{r-1}$ is

$$\partial_r \lambda = \sum_{j=1}^{r} \sum_{l=0}^{1} (-1)^{j+l} F^l_j(\lambda).$$
Let $\Lambda$ be a $k$-graph and let $G$ be an abelian group. For $r \geq 0$, let $C^r(\Lambda, G)$ be the collection of all functions $f : Q_r(\Lambda) \to G$.

Define maps $\delta^r : C^r(\Lambda, G) \to C^{r+1}(\Lambda, G)$ by

$$\delta^r(f)(\lambda) := f(\partial_{r+1}(\lambda)) = \sum_{j=1}^{r+1} \sum_{l=0}^{1} (-1)^{j+l} f(F^l_j(\lambda)).$$

Then $(C^\bullet(\Lambda, G), \delta^\bullet)$ is a cochain complex.

For $r \geq 0$, let $Z^r(\Lambda, G) := \ker(\delta^r)$ be the group of $r$-cocycles, and for $r > 0$, let $B^r(\Lambda, G) = \text{Im}(\delta^{r-1})$ be the group of $r$-coboundaries.

The cubical cohomology groups are

$$H^r(\Lambda, G) = Z^r(\Lambda, G)/B^r(\Lambda, G).$$
The long exact sequence of cohomology

- If $\Lambda$ is a row-finite $k$-graph with no sources and $\alpha \in \text{Aut} \, \Lambda$, then there is a $(k + 1)$-graph $\Lambda \times_\alpha \mathbb{Z}$ with morphisms $\Lambda \times \mathbb{N}$, range and source maps

  \[ r(\lambda, n) = (r(\lambda), 0), \quad s(\lambda, n) = (\alpha^{-n}(s(\lambda)), 0), \]

  degree map $d(\lambda, n) = (d(\lambda), n)$ and composition

  \[ (\lambda, m)(\mu, n) := (\lambda \alpha^m(\mu), m + n). \]

- **Theorem.** There is a long exact sequence

  \[ 0 \to H^0(\Lambda \times_\alpha \mathbb{Z}, G) \xrightarrow{i^*} H^0(\Lambda, G) \xrightarrow{1-\alpha^*} H^0(\Lambda, G) \xrightarrow{j^*} H^1(\Lambda \times_\alpha \mathbb{Z}, G) \xrightarrow{i^*} \cdots \]

  \[ \xrightarrow{1-\alpha^*} H^r(\Lambda, G) \xrightarrow{j^*} H^{r+1}(\Lambda \times_\alpha \mathbb{Z}, G) \xrightarrow{i^*} H^{r+1}(\Lambda, G) \xrightarrow{1-\alpha^*} H^{r+1}(\Lambda, G) \xrightarrow{j^*} \cdots \]

  \[ \xrightarrow{j^*} H^k(\Lambda \times_\alpha \mathbb{Z}, G) \xrightarrow{i^*} H^k(\Lambda, G) \xrightarrow{1-\alpha^*} H^k(\Lambda, G) \xrightarrow{j^*} H^{k+1}(\Lambda \times_\alpha \mathbb{Z}, G) \to 0. \]
Here
\[ i : \Lambda \to \Lambda \times_\alpha \mathbb{Z}, \quad i(\lambda) = (\lambda, 0) \quad \text{and} \quad j : \Lambda \times_\alpha \mathbb{Z} \to \Lambda, \quad j(\lambda, n) = \lambda \]
induce maps

\[ i^* : C^r(\Lambda \times_\alpha \mathbb{Z}, G) \to C^r(\Lambda, G), \quad i^*(f)(\lambda) = f(\lambda, 0) \]
and

\[ j^* : C^{r-1}(\Lambda, G) \to C^r(\Lambda \times_\alpha \mathbb{Z}, G), \]

\[ j^*(f)(\lambda, n) = \begin{cases} 0 & \text{if } n = 0 \\ f(\lambda) & \text{if } n = 1. \end{cases} \]
Twisted $k$-graph $C^*$-algebras

Let $\Lambda$ be a row-finite $k$-graph with no sources and let $\varphi \in Z^2(\Lambda, \mathbb{T})$. The twisted graph algebra $C^*_\varphi(\Lambda)$ is the universal $C^*$-algebra generated by a Cuntz-Krieger $\varphi$-representation of $\Lambda$.

A Cuntz-Krieger $\varphi$-representation of $\Lambda$ in a $C^*$-algebra $A$ is a set \( \{p_v : v \in \Lambda^0\} \) of mutually orthogonal projections and a set \( \{s_\lambda : \lambda \in Q_1(\Lambda)\} \) of partial isometries in $A$ satisfying

- for all $\lambda \in \Lambda^{e_i}$, $s_\lambda^* s_\lambda = p_{s(\lambda)}$;
- for all $1 \leq i < j \leq k$ and $\mu, \mu' \in \Lambda^{e_i}$, $\nu, \nu' \in \Lambda^{e_j}$ such that $\mu \nu = \nu' \mu'$,
  \[ s_{\nu'} s_{\mu'} = \varphi(\mu \nu) s_{\mu} s_{\nu} \]
- for all $v \in \Lambda^0$ we have
  \[ p_v = \sum_{\lambda \in v \Lambda^{e_i}} s_\lambda s_\lambda^*. \]

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Crossed products and twisted $k$-graph algebras
Theorem. Let $\Lambda$ be a row-finite $k$-graph with no sources, let $\alpha \in \text{Aut}(\Lambda)$ and let $\varphi \in Z^2(\Lambda \times \alpha \mathbb{Z}, \mathbb{T})$. Then

there is an automorphism $\alpha \varphi$ of $C^*_i(\varphi)(\Lambda)$ such that

$$\alpha \varphi(p_v) = p_{\alpha v} \text{ and } \alpha \varphi(s_e) = \varphi(\alpha e, 1)s_{\alpha e}$$

for all $v \in Q_0(\Lambda)$ and $e \in Q_1(\Lambda)$;

there is a homomorphism $\pi : C^*_i(\varphi)(\Lambda) \to C^*_\varphi(\Lambda \times \alpha \mathbb{Z})$ and an isomorphism $\pi \times U : C^*_i(\varphi)(\Lambda) \rtimes_{\alpha \varphi} \mathbb{Z} \to C^*_\varphi(\Lambda \times \alpha \mathbb{Z})$.

Here $U \in \mathcal{M}C^*_\varphi(\Lambda \times \alpha \mathbb{Z})$ satisfies

$$Up_{(v,0)}U^* = p_{(\alpha v,0)} \text{ and } Us_{(e,0)}U^* = \varphi(\alpha e, 1)s_{(\alpha e,0)}$$

for $v \in Q_0(\Lambda)$ and $e \in Q_1(\Lambda)$. 
Corollary 1. If \( \varphi = j^*(c) \) for some \( c \in \mathbb{Z}^1(\Lambda, \mathbb{T}) \), then

the automorphism \( \alpha^c := \alpha_\varphi \) of \( C^*(\Lambda) \) is given by

\[
\alpha^c(p_v) = p_{\alpha v} \quad \text{and} \quad \alpha^c(s_e) = c(\alpha e)s_{\alpha e}
\]

for \( v \in Q_0(\Lambda) \) and \( e \in Q_1(\Lambda) \);

there is an isomorphism \( \pi \times U : C^*(\Lambda) \rtimes_{\alpha^c} \mathbb{Z} \to C^*_{j^*(c)}(\Lambda \times \alpha \mathbb{Z}) \).

Corollary 2. Suppose that \( \psi \in \ker(1 - \alpha^*) \subseteq \mathbb{Z}^2(\Lambda, \mathbb{T}) \). Then

there is a function \( b \in C^1(\Lambda, \mathbb{T}) \) and there is an action \( \hat{\alpha} \) on \( C^*_{\psi}(\Lambda) \) given by

\[
\hat{\alpha}(p_v) = p_{\alpha v} \quad \text{and} \quad \hat{\alpha}(s_e) = b(\alpha e)s_{\alpha e}
\]

for all \( v \in Q_0(\Lambda) \) and \( e \in Q_1(\Lambda) \);

there is \( \varphi \in \mathbb{Z}^2(\Lambda \times \alpha \mathbb{Z}, \mathbb{T}) \) and an isomorphism

\[
\pi \times U : C_{\psi}^*(\Lambda) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C^*_\varphi(\Lambda \times \alpha \mathbb{Z}).
\]
Consider the 2-graph $\Lambda$ with two blue edges $f_0, f_1$ and three red edges $g_0, g_1, g_2$

subject to the relations

\[ f_0g_0 = g_1f_1, \quad f_1g_0 = g_1f_0, \quad f_0g_1 = g_2f_1, \]
\[ f_1g_1 = g_2f_0, \quad f_0g_2 = g_0f_1, \quad f_1g_2 = g_0f_0. \]

Then

- $H^0(\Lambda, G) \cong G$,
- $H^1(\Lambda, G) \cong G^2$,
- $H^2(\Lambda, G) \cong G^3$.

$C^*(\Lambda)$ is simple and purely infinite with $K_0(C^*(\Lambda)) = K_1(C^*(\Lambda)) = 0$, hence $C^*(\Lambda) \cong O_2$. 

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The permutations (01) and (012) induce an automorphism $\alpha$ of $\Lambda$.

Notice that $\alpha$ is periodic and $C^*(\Lambda) \rtimes_{\alpha} \mathbb{Z}$ is not simple. To get a simple algebra, we twist by a cocycle.

Fix $z, w \in \mathbb{T}$ such that $z^n, w^n \neq 1$ for all $n \in \mathbb{Z}$ and let $c \in Z^1(\Lambda, \mathbb{T})$ be such that

$$c(f_0) = c(f_1) = z, \quad c(g_0) = c(g_1) = c(g_2) = w.$$ 

We construct a representation of $C^*(\Lambda)$ on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ with basis $\{e_{i,q,k}\}_{i,k \geq 1, q=0,1}$. 
We take

\[ S_{f_0}(e_i, 0, k) = e_{2i-1}, 0, k, \quad S_{f_0}(e_i, 1, k) = e_{2i}, 1, k, \]
\[ S_{f_1}(e_i, 0, k) = e_{2i}, 0, k, \quad S_{f_1}(e_i, 1, k) = e_{2i-1}, 1, k, \]
\[ T_{g_m}(e_i, q, k) = e_{i, 1-q, 3k-2+m}, m = 0, 1, 2. \]

Let \( \alpha_c \) be the automorphism of \( C^*(\Lambda) \) such that

\[ \alpha_c(S_{f_i}) = zS_{f_{i+1} \mod 2}, \quad \alpha_c(T_{g_i}) = wT_{g_{i+1} \mod 3}. \]

Using an Archbold type technique we prove that \((\alpha_c)^n\) is outer for all \( n \).

It follows that \( C^*(\Lambda) \rtimes_{\alpha_c} \mathbb{Z} \cong C^*_{j^*(c)}(\Lambda \times_{\alpha} \mathbb{Z}) \) is simple and purely infinite.
To compute the cohomology of $\Lambda$ and of $\Lambda \times_\alpha \mathbb{Z}$, we first compute the homology $H_n(\Lambda)$, we determine the maps $\alpha_* : H_n(\Lambda) \to H_n(\Lambda)$ induced by $\alpha$ and use the homology exact sequence to compute $H_n(\Lambda \times_\alpha \mathbb{Z})$.

Then we apply the Universal Coefficient Theorem

$$0 \to \text{Ext}(H_{n-1}(\Lambda \times_\alpha \mathbb{Z}), G) \to H^n(\Lambda \times_\alpha \mathbb{Z}, G) \to \text{Hom}(H_n(\Lambda \times_\alpha \mathbb{Z}), G) \to 0$$

to compute $H^n(\Lambda \times_\alpha \mathbb{Z}, G)$.

We have

$$Q_0(\Lambda) = \{v\}, Q_1(\Lambda) = \{f_0, f_1, g_0, g_1, g_2\},$$

$$Q_2(\Lambda) = \{f_0g_0 = g_1f_1, f_1g_0 = g_1f_0, f_0g_1 = g_2f_1, f_1g_1 = g_2f_0, f_0g_2 = g_0f_1, f_1g_2 = g_0f_0\},$$

Moreover, we have $\partial_0 = \partial_1 = 0$ and

$$\partial_2(f_0g_0) = f_0 + g_0 - g_1 - f_1, \quad \partial_2(f_1g_0) = f_1 + g_0 - g_1 - f_0,$$

$$\partial_2(f_0g_1) = f_0 + g_1 - g_2 - f_1, \quad \partial_2(f_1g_1) = f_1 + g_1 - g_2 - f_0,$$

$$\partial_2(f_0g_2) = f_0 + g_2 - g_0 - f_1, \quad \partial_2(f_1g_2) = f_1 + g_2 - g_0 - f_0.$$
Using the Smith normal form of $\partial_2$ we see that $\ker \partial_2$ has generators

\begin{align*}
  b_1 &= f_0g_0 + f_1g_0 + 2f_1g_1 + 2f_0g_2, \\
  b_2 &= -2f_0g_0 + f_0g_1 - 3f_1g_1 - 2f_0g_2, \\
  b_3 &= 2f_0g_0 + 2f_1g_1 + f_0g_2 + f_1g_2
\end{align*}

and $\text{im } \partial_2$ has generators

\begin{align*}
  f_0 - f_1 + g_0 - g_1, \quad g_0 - g_2, \quad g_1 - g_2.
\end{align*}

It follows that

\begin{align*}
  H_0(\Lambda) &= \mathbb{Z}, \\
  H_1(\Lambda) &= \mathbb{Z}^5 / \text{im } \partial_2 \cong \mathbb{Z}^2, \\
  H_2(\Lambda) &= \ker \partial_2 \cong \mathbb{Z}^3
\end{align*}

and

\begin{align*}
  H^0(\Lambda, G) &\cong G, \\
  H^1(\Lambda, G) &\cong G^2, \\
  H^2(\Lambda, G) &\cong G^3.
\end{align*}
Since $\alpha(f_0) = f_1$, $\alpha(f_1) = f_0$, $\alpha(g_0) = g_1$, $\alpha(g_1) = g_2$, $\alpha(g_2) = g_0$ and $H_1(\Lambda)$ is generated by $[f_0]$, $[g_0]$, we see that $\alpha_* : H_1(\Lambda) \to H_1(\Lambda)$ is the identity map.

Since $\alpha(b_1) = 2b_1 + b_2$, $\alpha(b_2) = -2b_1 + b_3$, $\alpha(b_3) = b_1$, it follows that $\alpha_* : H_2(\Lambda) \to H_2(\Lambda)$ has matrix

$$
\begin{bmatrix}
2 & 1 & 0 \\
-2 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
$$

and $\ker(1 - \alpha_*) \cong \mathbb{Z}$, $\text{im}(1 - \alpha_*) \cong \mathbb{Z}^2$.

Using the long exact sequence of homology we get

$$
H_1(\Lambda \times_\alpha \mathbb{Z}) \cong \mathbb{Z}^3, \quad H_2(\Lambda \times_\alpha \mathbb{Z}) \cong \mathbb{Z}^3, \quad H_3(\Lambda \times_\alpha \mathbb{Z}) \cong \mathbb{Z}.
$$

It follows that

$$
H^1(\Lambda \times_\alpha \mathbb{Z}, G) \cong G^3, \quad H^2(\Lambda \times_\alpha \mathbb{Z}, G) \cong G^3, \quad H^3(\Lambda \times_\alpha \mathbb{Z}, G) \cong G.
$$
References


