2.1 Let \( \{X_t\} \) be a sequence of uncorrelated random variables, each with mean 0 and variance \( \sigma^2 \). For each of the following processes, find its representation in terms of lagged \( X \)-values (i.e. \( X_{t+k}, k = 0, 1, 2, \ldots \)), in terms of powers of the backshift operator \( B \), find weights of a corresponding linear filter \( W_t = \sum w_j X_{t-j} \), find the mean and variance of \( W_t \):

a) \( W_t = \nabla X_t \):
\[
W_t = (1 - B)X_t = X_t - X_{t-1}
\]
\[
W_t = \sum_{j=0}^1 w_j X_{t-j}, \text{ where } \{w_j\} = \{1, -1\}.
\]

b) \( W_t = \nabla^2 X_t \):
\[
W_t = (1 - B)^2 X_t = (1 - 2B + B^2)X_t
\]
\[
W_t = X_t - 2X_{t-1} + X_{t-2}
\]
\[
W_t = \sum_{j=0}^2 w_j X_{t-j}, \text{ where } \{w_j\} = \{1, -2, 1\}.
\]

c) \( W_t = \nabla^3 X_t \):
\[
W_t = (1 - B^3) X_t = (1 - B^3)X_t = X_t - X_{t-3}
\]
\[
W_t = \sum_{j=0}^3 w_j X_{t-j}, \text{ where } \{w_j\} = \{1, 0, 0, -1\}.
\]

d) \( W_t = \nabla^2 \nabla^3 X_t \):
\[
W_t = (1 - B)^2 (1 - B^3)X_t
\]
\[
W_t = (1 - 2B + B^2 - B^3 + 2B^4 - B^5)X_t
\]
\[
W_t = X_t - 2X_{t-1} + X_{t-2} - X_{t-3} + 2X_{t-4} - X_{t-5}
\]
\[
W_t = \sum_{j=0}^5 w_j X_{t-j}, \text{ where } \{w_j\} = \{1, -2, 1, -1, 2, -1\}.
\]

2.2 Find all non-trivial (not equal to identity) linear filters of the form \( \alpha + \beta B + \gamma B^2 \) (i.e. find \( \alpha, \beta, \) and \( \gamma \)) that do not change the variance and mean of uncorrelated sequences.

**Solution:** Let \( X_t \) be an uncorrelated sequence of random variables. Applying a linear filter of the form \( \alpha + \beta B + \gamma B^2 \) yields a new time series \( Y_t \):
\[
Y_t = (\alpha + \beta B + \gamma B^2)X_t = \alpha X_t + \beta X_{t-1} + \gamma X_{t-2}
\]
We begin by examining the requirement that the mean of $Y_t$ is unchanged

\[ E(X_t) = E(Y_t) \]
\[ = E(\alpha X_t + \beta X_{t-1} + \gamma X_{t-2}) \]
\[ = E(\alpha X_t) + E(\beta X_{t-1}) + E(\gamma X_{t-2}) \]
\[ = \alpha E(X_t) + \beta E(X_{t-1}) + \gamma E(X_{t-2}) \]
\[ = (\alpha + \beta + \gamma)E(X_t) \]

In which case, the coefficients of the linear filter must satisfy the following relationship $\alpha + \beta + \gamma = 1$. Next we consider the variance, which also remains unchanged, such that

\[ \text{Var}(X_t) = \text{Var}(Y_t) \]
\[ = \text{Var}(\alpha X_t + \beta X_{t-1} + \gamma X_{t-2}) \]
\[ = \text{Var}(\alpha X_t) + \text{Var}(\beta X_{t-1}) + \text{Var}(\gamma X_{t-2}) \]
\[ = \alpha^2 \text{Var}(X_t) + \beta^2 \text{Var}(X_{t-1}) + \gamma^2 \text{Var}(X_{t-2}) \]
\[ = (\alpha^2 + \beta^2 + \gamma^2) \text{Var}(X_t) \]

Thus the coefficients of the linear filter must also satisfy the condition $\alpha^2 + \beta^2 + \gamma^2 = 1$.

Finally, the solution is the set of all triplets $(\alpha, \beta, \gamma)$, excluding $(\alpha = 1, \beta = \gamma = 0)$, obtained as the intersection of a three-dimensional unit sphere and a plane $\alpha + \beta + \gamma = 1$. For example, the following triplet is a valid solutions: $(\alpha = \gamma = 0, \beta = 1)$.

2.3 Let $\{Z_t\}$ be a sequence of independent normal random variables, each with mean 0 and variance $\sigma^2$, and let $a$, $b$, $c$ be constants. For each process below decide whether it is stationary or not. For each stationary process find the mean, autocovariance function, and autocorrelation function:

**Solution:**

1) $X_t = a + bZ_t + cZ_{t+1}$

(i) $E(X_t) = \mu$:

\[ E(X_t) = E(a + bZ_t + cZ_{t+1}) \]
\[ = a + bE(Z_t) + cE(Z_{t+1}) \]
\[ = a \quad \checkmark \]
(ii) $\gamma_X(h) = f(h)$:

$$\gamma_X(h) = E\{[X_t - \mu][X_{t+h} - \mu]\}$$
$$= E\{[a + bZ_t + cZ_{t+1} - a][a + bZ_{t+h} + cZ_{t+1+h} - a]\}$$
$$= E\{b^2Z_tZ_{t+h} + bc[Z_tZ_{t+h+1} + Z_{t+1}Z_{t+h}] + c^2Z_{t+1}Z_{t+h+1}\}$$
$$= b^2E(Z_tZ_{t+h}) + bcE(Z_{t+1}Z_{t+h}) + c^2E(Z_{t+1}Z_{t+h+1})$$
$$= \begin{cases} (b^2 + c^2)\sigma^2, & h = 0 \\ bc\sigma^2, & h = 1 \\ 0, & h \geq 2 \end{cases}$$

This time series meets the criteria for a weakly stationary process, with mean and autocovariance as shown above and the following ACF:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1, & h = 0 \\ \frac{bc}{b^2+c^2}, & h = 1 \\ 0, & h \geq 2 \end{cases}$$

2) $X_t = a + bZ_0$

(i) $E(X_t) = \mu$:

$$E(X_t) = E(a + bZ_0)$$
$$= a + bE(Z_0)$$
$$= a$$

(ii) $\gamma_X(h) = f(h)$:

$$\gamma_X(h) = E\{[X_t - \mu][X_{t+h} - \mu]\}$$
$$= E\{[a + bZ_0 - a][a + bZ_0 - a]\}$$
$$= E(b^2Z_0^2)$$
$$= b^2\sigma^2$$

This time series meets the criteria for a weakly stationary process, with mean $a$, covariance $b^2\sigma^2$ and an ACF of 1 at lag $h = 0$.

3) $X_t = Z_tZ_{t-1}$

(i) $E(X_t) = \mu$:

$$E(X_t) = E(Z_tZ_{t-1})$$
$$= 0$$

3
(ii) $\gamma_X(h) = f(h)$:

$$\gamma_X(h) = E\{[X_t - \mu][X_{t+h} - \mu]\}$$
$$= E\{[Z_tZ_{t-1} - 0][Z_{t+h}Z_{t+h-1} - 0]\}$$
$$= E(Z_tZ_{t-1}Z_{t+h}Z_{t+h-1})$$
$$= \begin{cases} \sigma^4, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

This time series meets the criteria for a weakly stationary process, with mean and autocovariance as shown above and the following ACF:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

2.4 Let $\{Z_t\}$ be a stationary process with mean zero and let $a$ and $b$ be constants. If $X_t = a + bt + S_t + Z_t$, where $S_t$ is a seasonal component with period $d$, show that process $\nabla_dX_t$ is stationary and express its autocovariance function in terms of that of $\{Z_t\}$.

**Solution:** We can express the process $Y_t = \nabla_dX_t$ in terms of the backshift operator $B$:

$$Y_t = \nabla_dX_t = (1 - B)(1 - B^d)X_t$$
$$= (1 - B - B^d + B^{d+1})X_t$$

We can then rewrite this equation in terms of process $Z_t$

$$Y_t = a + bt + S_t + Z_t$$
$$- a - b(t - 1) - S_{t-1} - Z_{t-1}$$
$$- a - b(t - d) - S_{t-d} - Z_{t-d}$$
$$+ a + b(t - d - 1) + S_{t-d-1} + Z_{t-d-1}$$
$$= Z_t - Z_{t-1} - Z_{t-d} + Z_{t-d-1}.$$
(i) $E(Y_t) = \mu$:

$$E(Y_t) = E(Z_t - Z_{t-1} - Z_{t-d} + Z_{t-d-1})$$
$$= E(Z_t) - E(Z_{t-1}) - E(Z_{t-d}) + E(Z_{t-d-1})$$
$$= 4E(Z_t)$$
$$= 0 \checkmark$$

(ii) $\gamma_Y(h) = f(h)$:

$$\gamma_Y(h) = E\{[Y_t - \mu][Y_{t+h} - \mu]\}$$
$$= E\{[Y_t - 0][Y_{t+h} - 0]\}$$
$$= E(Y_t Y_{t+h})$$

$$= E\{[Z_t - Z_{t-1} - Z_{t-d} + Z_{t-d-1}][Z_{t+h} - Z_{t+h-1} - Z_{t+h-d} + Z_{t+h-d-1}]\}$$
$$= E\{Z_t Z_{t+h} - Z_t Z_{t+h-1} - Z_t Z_{t+h-d} + Z_t Z_{t+h-d-1}$$
$$- Z_{t-1} Z_{t+h} + Z_{t-1} Z_{t+h-1} + Z_{t-1} Z_{t+h-d} - Z_{t-1} Z_{t+h-d-1}$$
$$- Z_{t-d} Z_{t+h} + Z_{t-d} Z_{t+h-1} + Z_{t-d} Z_{t+h-d} - Z_{t-d} Z_{t+h-d-1}$$
$$+ Z_{t-d-1} Z_{t+h} - Z_{t-d-1} Z_{t+h-1} - Z_{t-d-1} Z_{t+h-d} + Z_{t-d-1} Z_{t+h-d-1}\}.$$ 

Assuming $d \geq 1$, this simplifies to

$$\gamma_Y(h) = \begin{cases} 
\gamma_Y(-h), & h \leq -1 \\
4\gamma_Z(0), & h = 0, d \neq 1 \\
5\gamma_Z(0), & h = 0, d = 1 \\
-4\gamma_Z(0), & h = d = 1 \\
-2\gamma_Z(0), & h = 1 \text{ or } h = d, d \neq 1 \\
\gamma_Z(0), & h = d \pm 1 \\
0, & \text{o.w.} 
\end{cases} \checkmark$$